Effective non-vanishing of global sections of multiple adjoint bundles for polarized 4-folds [*][†][‡]

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Abstract

Let X be a smooth complex projective variety of dimension 4 and let L be an ample line bundle on X. In this paper, we study a natural number m such that $h^0(m(K_X + L)) > 0$ for any polarized 4-folds (X, L) with $\kappa(K_X + L) \geq 0$.

1 Introduction

Let X be a smooth projective variety of dimension n defined over the field of complex numbers and let L be an ample (resp. nef and big) line bundle on X. Then the pair (X, L) is called a polarized (resp. quasi-polarized) manifold.

Then there are the following conjectures.

Conjecture 1 (i) (**Ionescu [25, Open problems, P.321]**) *Let (X, L) be a quasi-polarized manifold of dimension n. Assume that $K_X + L$ is nef. Then $h^0(K_X + L) > 0$.*

(ii) (**Ambro [1, Section 4], Kawamata [22, Conjecture 2.1]**) *Let X be a complex normal variety, B an effective $\mathbb{R}$-divisor on X such that the pair (X, B) is KLT, and D a Cartier divisor on X. Assume that D is nef, and that $D - (K_X + B)$ is nef and big. Then $h^0(D) > 0$.*

These conjectures have been studied by several authors (see [22], [8], [15], [18], [6], [7], [21]). In particular it is known that Conjecture 1 (i) is true if $\dim X \leq 3$, and Conjecture 1 (ii) is true if $\dim X \leq 2$.

Here we note that if $K_X + L$ is nef, then by [26] there exists a positive integer m such that $h^0(m(K_X + L)) > 0$, that is, $\kappa(K_X + L) \geq 0$. So, as a generalization of Conjecture 1 (i), it is natural and interesting to study the following problem, which was proposed in [15, Problem 3.2]:

Problem 1 *For any fixed positive integer n, determine the smallest positive integer p, which depends only on n, such that the following $(*)$ is satisfied:*

$(*)$ $h^0(p(K_X + L)) > 0$ for any polarized manifold (X, L) of dimension n with $\kappa(K_X + L) \geq 0$.

The aim of this paper is to study Problem 1. It is known that $p = 1$ if X is a curve or surface (see [15, Theorem 2.8]). For the case of $n = 3$, in [18, Theorems 5.1 and 5.2], we proved $p \leq 2$. Concretely, in [17, Theorem 5.4 (2)] or [18, Theorem 5.2], we proved that if $\kappa(K_X + L) = 3$, then $h^0(2(K_X + L)) \geq 3$. Moreover in [18, Theorem 5.1], we proved that $h^0(K_X + L) > 0$ if $0 \leq \kappa(K_X + L) \leq 2$. Moreover by using a result of Höring ([21, 1.5 Theorem]) and the adjunction theory, we can get $p = 1$ if $\dim X = 3$. Therefore Problem 1 was completly solved for the case of $\dim X = 3$.

So, as the next step, in this paper, we will treat the case of $\dim X = 4$. The main result of this paper is the following.

Theorem 1 *Let (X, L) be a polarized manifold of dimension 4. Assume that $\kappa(K_X + L) \geq 0$ holds.*

(a) *If $0 \leq \kappa(K_X + L) \leq 2$, then $h^0(2m(K_X + L)) > 0$ holds for every integer m with $m \geq 1$. In particular $h^0(2(K_X + L)) > 0$.*

(b) *If $\kappa(K_X + L) = 3$, then $h^0(2m(K_X + L)) > 0$ holds for every integer m with $m \geq 2$. In particular $h^0(4(K_X + L)) > 0$.*

(c) *If $\kappa(K_X + L) = 4$, then $h^0(2m(K_X + L)) > 0$ holds for every integer m with $m \geq 3$. In particular $h^0(6(K_X + L)) > 0$.*

(d) *If $\kappa(X) \geq 0$, then*

$$h^0(2m(K_X + L)) \geq \frac{(m-1)(m-2)(m^2 + 3m + 6)}{12} + 1$$

for every integer $m \geq 2$. In particular $h^0(4(K_X + L)) > 0$.

By using the adjunction theory of Beltrametti and Sommese, Theorem 1 is obtained from the following result, which will be proved in this paper (see Theorems 3.2, 3.3, 3.4, 3.5 and 3.6).

Theorem 2 *Let (X, L) be a polarized variety of dimension 4 such that X is a normal Gorenstein projective variety with only isolated terminal singularities. Assume that $K_X + L$ is nef.*

(a) *If $0 \leq \kappa(K_X + L) \leq 2$, then $h^0(m(K_X + L)) > 0$ for every integer m with $m \geq 1$.*

(b) *If $\kappa(K_X + L) = 3$, then $h^0(m(K_X + L)) > 0$ for every integer m with $m \geq 2$.*

(c) *If $\kappa(K_X + L) = 4$, then $h^0(m(K_X + L)) > 0$ for every integer m with $m \geq 3$.*

(d) *If $\kappa(X) \geq 0$, then*

$$h^0(m(K_X + L)) \geq \frac{(m-1)(m-2)(m^2 + 3m + 6)}{12} + 1$$

for every integer $m \geq 2$. In particular $h^0(2(K_X + L)) > 0$.

Here we note that recently Arakawa [2, Theorem 1.5] proved that $h^0(m(K_X + L)) > 0$ for any polarized n-folds (X, L) such that $K_X + L$ is nef, and for every integer m with $m \geq (n(n+1)/2) + 2$.

In this paper, we shall study mainly a smooth projective variety X over the field of complex numbers $\mathbb{C}$. We will employ the customary notation in algebraic geometry.

2   Preliminaries

Definition 2.1 (1) Let X (resp. Y) be an n-dimensional projective manifold, and let $\mathcal{L}$ (resp. $\mathcal{A}$) be an ample line bundle on X (resp. Y). Then $(X, \mathcal{L})$ is called a *simple blowing up of* $(Y, \mathcal{A})$ if there exists a birational morphism $\pi : X \to Y$ such that π is a blowing up at a point of Y and $\mathcal{L} = \pi^*(\mathcal{A}) - E$, where E is the exceptional divisor.
(2) Let X (resp. M) be an n-dimensional projective manifold, and let $\mathcal{L}$ (resp. $\mathcal{A}$) be an ample line bundle on X (resp. M). Then we say that $(M, \mathcal{A})$ is a *reduction of* $(X, \mathcal{L})$ if $(X, \mathcal{L})$ is obtained by a composite of simple blowing ups of $(M, \mathcal{A})$, and $(M, \mathcal{A})$ is not obtained by a simple blowing up of any polarized manifold. The morphism $\mu : X \to M$ is called the *reduction map*.

Definition 2.2 Let $(X, \mathcal{L})$ be a polarized manifold of dimension n. We say that $(X, \mathcal{L})$ is a *scroll* (resp. *quadric fibration, Del Pezzo fibration*) *over a normal projective variety* Y *with* $\dim Y = m$ if there exists a surjective morphism with connected fibers $f : X \to Y$ such that $K_X + (n-m+1)\mathcal{L} = f^*\mathcal{A}$ (resp. $K_X + (n-m)\mathcal{L} = f^*\mathcal{A}$, $K_X + (n-m-1)\mathcal{L} = f^*\mathcal{A}$) for some ample line bundle $\mathcal{A}$ on Y.

Remark 2.1 If $(X, \mathcal{L})$ is a scroll over a smooth curve C (resp. a smooth projective surface S) with $\dim X = n \geq 3$, then by [5, (3.2.1) Theorem] and [4, Proposition 3.2.1] there exists an ample vector bundle $\mathcal{E}$ of rank n (resp. $n-1$) on C (resp. S) such that $(X, \mathcal{L}) \cong (\mathbb{P}_C(\mathcal{E}), H(\mathcal{E}))$ (resp. $(\mathbb{P}_S(\mathcal{E}), H(\mathcal{E})))$.

Here we give the definition of the ith sectional geometric genus of multi-prepolarized varieties.

Notation 2.1 Let X be a projective variety of dimension n, let i be an integer with $0 \leq i \leq n-1$, let $L_1, \ldots, L_{n-i}$ be line bundles on X and let $\mathcal{F}$ be a coherent sheaf on X. Then $\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}} \otimes \mathcal{F})$ is a polynomial in $t_1, \ldots, t_{n-i}$ of total degree at most n. So we can write $\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}} \otimes \mathcal{F})$ uniquely as follows.

$$\chi(L_1^{t_1} \otimes \cdots \otimes L_{n-i}^{t_{n-i}} \otimes \mathcal{F})$$
$$= \sum_{p=0}^{n} \sum_{\substack{p_1 \geq 0, \ldots, p_{n-i} \geq 0 \\ p_1 + \cdots + p_{n-i} = p}} \chi_{p_1, \ldots, p_{n-i}}(L_1, \ldots, L_{n-i}; \mathcal{F}) \binom{t_1 + p_1 - 1}{p_1} \cdots \binom{t_{n-i} + p_{n-i} - 1}{p_{n-i}}.$$

Definition 2.3 Let X be a projective variety of dimension n, let i be an integer with $0 \leq i \leq n-1$, let $L_1, \ldots, L_{n-i}$ be line bundles on X and let $\mathcal{F}$ be a coherent sheaf on X.
(1) For every $i \in \mathbb{Z}$ with $0 \leq i \leq n$, we set

$$\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathcal{F}) := \begin{cases} \underbrace{\chi_{1, \ldots, 1}}_{n-i}(L_1, \ldots, L_{n-i}; \mathcal{F}) & \text{if } 0 \leq i \leq n-1, \\ \chi(\mathcal{F}) & \text{if } i = n. \end{cases}$$

(2) For every $i \in \mathbb{Z}$ with $0 \leq i \leq n$, the *ith sectional geometric genus* $g_i(X, L_1, \ldots, L_{n-i}; \mathcal{F})$ is defined by the following:

$$\begin{aligned} g_i(X, L_1, \ldots, L_{n-i}; \mathcal{F}) &= (-1)^i (\chi_i^H(X, L_1, \ldots, L_{n-i}) - \chi(\mathcal{F})) \\ &\quad + \sum_{j=0}^{n-i} (-1)^{n-i-j} h^{n-j}(\mathcal{F}). \end{aligned}$$

Definition 2.4 Let X be a projective variety of dimension n, let i be an integer with $0 \leq i \leq n-1$, and let $L_1, \ldots, L_{n-i}$ be line bundles on X. Then we set

$$\begin{aligned} \chi_i^H(X, L_1, \ldots, L_{n-i}) &:= \chi_i^H(X, L_1, \ldots, L_{n-i}; \mathcal{O}_X), \\ g_i(X, L_1, \ldots, L_{n-i}) &:= g_i(X, L_1, \ldots, L_{n-i}; \mathcal{O}_X), \\ p_a^i(X, L_1, \ldots, L_{n-i}) &:= p_a^i(X, L_1, \ldots, L_{n-i}; \mathcal{O}_X). \end{aligned}$$

Remark 2.2 (1) We can prove that $\chi_{p_1, \ldots, p_{n-i}}(L_1, \ldots, L_{n-i}; \mathcal{F})$ is an integer for every nonnegative integers $p_1, \ldots, p_{n-i}$ with $0 \leq p_1 + \cdots + p_{n-i} \leq n$. So in particular we see that $g_i(X, L_1, \ldots, L_{n-i}; \mathcal{F})$ is an integer.

(2) If $i = 0$, then $g_0(X, L_1, \ldots, L_n) = L_1 \cdots L_n$.

(3) If $L_1 = \cdots = L_{n-i} = L$, then $g_i(X, \underbrace{L, \ldots, L}_{n-i}) = g_i(X, L)$. (Here $g_i(X, L)$ denotes the ith sectional geometric genus of (X, L) (see [13, Definition 2.1]).) In particular, if $i = n - 1$, then $g_{n-1}(X, L_1)$ in Definition 2.4 is equal to the $(n-1)$th sectional geometric genus of (X, L_1) in [13, Definition 2.1].

(4) If $i = n$, then $g_n(X) = h^n(\mathcal{O}_X)$.

(5) $\chi_i^H(X, L_1, \ldots, L_{n-i}; \mathcal{F})$ in Definition 2.3 (1) is called the ith sectional H-arithmetic genus of $(X, L_1, \ldots, L_{n-i}; \mathcal{F})$.

(6) Let X be a *smooth* projective variety of dimension n and let $\mathcal{E}$ be an ample vector bundle of rank r on X with $1 \leq r \leq n$. Then in [12, Definition 2.1], we defined the ith c_r-*sectional geometric genus* $g_i(X, \mathcal{E})$ of $(X, \mathcal{E})$ for every integer i with $0 \leq i \leq n - r$. Let i be an integer with $0 \leq i \leq n - 1$ and let $L_1, \ldots, L_{n-i}$ be ample line bundles on X. By setting $\mathcal{E} := L_1 \oplus \cdots \oplus L_{n-i}$, we see that $g_i(X, \mathcal{E}) = g_i(X, L_1, \ldots, L_{n-i})$.

Proposition 2.1 *Let X be a projective variety of dimension n and let i be an integer with $0 \leq i \leq n - 1$. Let $A, B, L_1, \cdots, L_{n-i-1}$ be line bundles on X. Then*

$$g_i(X, A + B, L_1, \cdots, L_{n-i-1})$$
$$= g_i(X, A, L_1, \cdots, L_{n-i-1}) + g_i(X, B, L_1, \cdots, L_{n-i-1})$$
$$+ g_{i-1}(X, A, B, L_1, \cdots, L_{n-i-1}) - h^{i-1}(\mathcal{O}_X).$$

Proof. See [16, Corollary 2.4 and Remark 2.6]. □

In [17, Theorem 5.1] we obtained the following equality under the assumption that X is smooth. But by the same argument as in the proof of [17], we can also prove this equality if X is a normal Gorenstein projective variety with $\dim X = n \geq 2$ such that X has at most terminal singularities because the Serre duality and the Kawamata-Viehweg vanishing theorem hold in this case.

Theorem 2.1 *Let X be a normal Gorenstein projective variety with $\dim X = n \geq 2$ such that X has at most terminal singularities, let $L_1, \cdots, L_m$ be nef and big line bundles on X and let L be a nef line bundle, where $m \geq 1$. Then*

$$h^0(K_X + L_1 + \cdots + L_m + L) - h^0(K_X + L_1 + \cdots + L_m)$$
$$= \sum_{s=0}^{n-1} \sum_{(k_1, \cdots, k_{n-s-1}) \in A_{n-s-1}^m} g_s(X, L_{k_1}, \cdots, L_{k_{n-s-1}}, L)$$
$$- \sum_{s=0}^{n-2} \binom{m-1}{n-s-2} h^s(\mathcal{O}_X).$$

Here $A_t^p := \{(k_1, \cdots, k_t) \mid k_l \in \{1, \cdots, p\}, k_i < k_j \text{ if } i < j\}$, and we set

$$\sum_{(k_1, \cdots, k_{n-s-1}) \in A_{n-s-1}^m} g_s(X, L_{k_1}, \cdots, L_{k_{n-s-1}}, L) = \begin{cases} 0 & \text{if } n - s - 1 > m, \\ g_{n-1}(X, L) & \text{if } s = n - 1. \end{cases}$$

Definition 2.5 Let $(\sharp)$ be an assumption of polarized varieties (X, L). For any fixed positive integer n, we set

$$\mathcal{P}_n(\sharp) := \{ (X, L) : \text{polarized variety} \mid \dim X = n, (X, L) \text{ satisfies } (\sharp)$$
$$\text{and } \kappa(K_X + L) \geq 0 \},$$

$$\mathcal{P}_n^{\mathrm{NEF}}(\sharp) \quad := \quad \{\,(X,L) : \text{polarized variety} \mid \dim X = n,\ (X,L) \text{ satisfies } (\sharp)$$
$$\text{and } K_X + L \text{ is nef}\,\},$$

$$\mathcal{M}_n(\sharp) \quad := \quad \{\,r \in \mathbb{N} \mid h^0(r(K_X + L)) > 0 \text{ for any } (X,L) \in \mathcal{P}_n(\sharp)\,\},$$
$$\mathcal{M}_n(\sharp)^+ \quad := \quad \{\,r \in \mathbb{N} \mid h^0(m(K_X + L)) > 0 \text{ for any } m \geq r \text{ and any } (X,L) \in \mathcal{P}_n(\sharp)\,\},$$
$$\mathcal{M}_n^{\mathrm{NEF}}(\sharp) \quad := \quad \{\,r \in \mathbb{N} \mid h^0(r(K_X + L)) > 0 \text{ for any } (X,L) \in \mathcal{P}_n^{\mathrm{NEF}}(\sharp)\,\},$$
$$\mathcal{M}_n^{\mathrm{NEF}}(\sharp)^+ \quad := \quad \{\,r \in \mathbb{N} \mid h^0(m(K_X + L)) > 0 \text{ for any } m \geq r \text{ and any } (X,L) \in \mathcal{P}_n^{\mathrm{NEF}}(\sharp)\,\},$$

$$m_n(\sharp) \quad := \quad \begin{cases} \min \mathcal{M}_n(\sharp) & \text{if } \mathcal{M}_n(\sharp) \neq \emptyset, \\ \infty & \text{if } \mathcal{M}_n(\sharp) = \emptyset. \end{cases}$$
$$m_n(\sharp)^+ \quad := \quad \begin{cases} \min \mathcal{M}_n(\sharp)^+ & \text{if } \mathcal{M}_n(\sharp)^+ \neq \emptyset, \\ \infty & \text{if } \mathcal{M}_n(\sharp)^+ = \emptyset. \end{cases}$$
$$m_n^{\mathrm{NEF}}(\sharp) \quad := \quad \begin{cases} \min \mathcal{M}_n^{\mathrm{NEF}}(\sharp) & \text{if } \mathcal{M}_n^{\mathrm{NEF}}(\sharp) \neq \emptyset, \\ \infty & \text{if } \mathcal{M}_n^{\mathrm{NEF}}(\sharp) = \emptyset. \end{cases}$$
$$m_n^{\mathrm{NEF}}(\sharp)^+ \quad := \quad \begin{cases} \min \mathcal{M}_n^{\mathrm{NEF}}(\sharp)^+ & \text{if } \mathcal{M}_n^{\mathrm{NEF}}(\sharp)^+ \neq \emptyset, \\ \infty & \text{if } \mathcal{M}_n^{\mathrm{NEF}}(\sharp)^+ = \emptyset. \end{cases}$$

Remark 2.3 Here we note that the following inequality hold.

$$\begin{aligned}
m_n(\sharp) &\leq m_n(\sharp)^+, \\
m_n^{\mathrm{NEF}}(\sharp) &\leq m_n^{\mathrm{NEF}}(\sharp)^+, \\
m_n^{\mathrm{NEF}}(\sharp) &\leq m_n(\sharp), \\
m_n^{\mathrm{NEF}}(\sharp)^+ &\leq m_n(\sharp)^+.
\end{aligned}$$

Lemma 2.1 *Let X be a complete normal variety of dimension n, and let D_1 and D_2 be effective Cartier divisors on X. Then $h^0(D_1 + D_2) \geq h^0(D_1) + h^0(D_2) - 1$.*

Proof. See [13, Lemma 1.12] or [24, 15.6.2 Lemma]. □

Lemma 2.2 *Let p and q be positive integers such that p and q are coprime. Then for any integer l with $l \geq (p-1)(q-1)$, there exist non-negative integers i and j such that $l = pi + qj$.*

Proof. We note that there exists a pair of integers (α, β) such that $p\alpha + q\beta = l$. Then we can easily see that any integers x and y which satisfy $px + qy = l$ can be expressed as $x = \alpha + qm$ and $y = \beta - pm$, where m is an arbitrary integer. In particular there exists a pair of integers (x_1, y_1) with $px_1 + qy_1 = l$ and $0 \leq x_1 < q$. If $y_1 \geq 0$, then we get the assertion. So we may assume that $y_1 < 0$. Then $l = px_1 + qy_1 \leq px_1 - q \leq p(q-1) - q = (p-1)(q-1) - 1$. But this is a contradiction because we assume that $l \geq (p-1)(q-1)$. □

Lemma 2.3 *Let X be a smooth projective variety of dimension $n \geq 4$ and let V be a normal projective variety of dimension $n \geq 4$ with $\dim \mathrm{Sing}(V) \leq n - 4$. Let $\pi : X \to V$ be a birational morphism such that $X \backslash \pi^{-1}(\mathrm{Sing}(V)) \cong V \backslash \mathrm{Sing}(V)$. Let E be a π-exceptional irreducible and reduced divisor on X, A_1 and A_2 line bundles on X and $L_1, \ldots, L_{n-3}$ line bundles on V. Then $EA_1A_2(\pi^*(L_1)) \cdots (\pi^*(L_{n-3})) = 0$.*

Proof. By [23, Proposition 4 in section 2, chapter I], we have

$$EA_1A_2(\pi^*(L_1)) \cdots (\pi^*(L_{n-3})) = (A_1|_E)(A_2|_E)(\pi^*(L_1))|_E \cdots (\pi^*(L_{n-3}))|_E.$$

On the other hand, since $\dim \mathrm{Sing} V \leq n-4$, we have $\dim \pi(E) \leq n-4$. Here we set $Z := \pi(E)$. Then

$$(A_1|_E)(A_2|_E)(\pi^*(L_1))|_E \cdots (\pi^*(L_{n-3}))|_E = (A_1|_E)(A_2|_E)((\pi|_E)^*(L_1|_Z)) \cdots ((\pi|_E)^*(L_{n-3}|_Z)).$$

Here we set

$$f(t_1,\ldots,t_{n-1}) := \chi(E, ((\pi|_E)^*(L_1|_Z))^{\otimes t_1} \otimes \cdots \otimes ((\pi|_E)^*(L_{n-3}|_Z))^{\otimes t_{n-3}} \otimes (A_1|_E)^{\otimes t_{n-2}} \otimes (A_2|_E)^{\otimes t_{n-1}}).$$

Then $f(t_1,\ldots,t_{n-1})$ is a polynomial of $t_1,\ldots,t_{n-1}$ of degree at most $n-1$. Let C_1 (resp. C_2, C_3, C_4, C_5, C_6) be the coefficient of $t_1 \cdots t_{n-3}$ (resp. $t_1 \cdots t_{n-3} t_{n-2}$, $t_1 \cdots t_{n-3} t_{n-1}$, $t_1 \cdots t_{n-3} t_{n-2}^2$, $t_1 \cdots t_{n-3} t_{n-1}^2$, $t_1 \cdots t_{n-3} t_{n-2} t_{n-1}$) in $f(t_1,\ldots,t_{n-1})$.
Then $f(t_1,\ldots,t_{n-3},0,0) = \chi(E, ((\pi|_E)^*(L_1|_Z))^{\otimes t_1} \otimes \cdots \otimes ((\pi|_E)^*(L_{n-3}|_Z))^{\otimes t_{n-3}})$. Here we set

$$g(t_1,\ldots,t_{n-3}) := f(t_1,\ldots,t_{n-3},0,0).$$

Then the coefficient of $t_1 \cdots t_{n-3}$ in $g(t_1,\ldots,t_{n-3})$ is equal to C_1. On the other hand since the degree of $g(t_1,\ldots,t_{n-3})$ is less than $n-3$ (see the proof of [23, Proposition 6 in section 2, chapter I]), we have $C_1 = 0$.

Next we consider the polynomial $f(t_1,\ldots,t_{n-3},1,0)$ (resp. $f(t_1,\ldots,t_{n-3},0,1)$). Then the coefficient of $t_1 \cdots t_{n-3}$ in $f(t_1,\ldots,t_{n-3},1,0)$ (resp. $f(t_1,\ldots,t_{n-3},0,1)$) is $C_1 + C_2 + C_4$ (resp. $C_1 + C_3 + C_5$). Moreover

$$f(t_1,\ldots,t_{n-3},1,0) = \chi(E, ((\pi|_E)^*(L_1|_Z))^{\otimes t_1} \otimes \cdots \otimes ((\pi|_E)^*(L_{n-3}|_Z))^{\otimes t_{n-2}} \otimes (A_1|_E))$$

and the degree of this polynomial is less than $n-3$. Hence $C_1 + C_2 + C_4 = 0$. By the same reason as this, we have $C_1 + C_3 + C_5 = 0$. Therefore $C_2 + C_4 = C_3 + C_5 = 0$ since $C_1 = 0$.

Finally we consider $f(t_1,\ldots,t_{n-3},1,1)$. Then the coefficient of $t_1 \cdots t_{n-3}$ in $f(t_1,\ldots,t_{n-3},1,1)$ is $C_1 + C_2 + C_3 + C_4 + C_5 + C_6$. Moreover

$$f(t_1,\ldots,t_{n-3},1,1) = \chi(E, ((\pi|_E)^*(L_1|_Z))^{\otimes t_1} \otimes \cdots \otimes ((\pi|_E)^*(L_{n-3}|_Z))^{\otimes t_{n-2}} \otimes (A_1|_E) \otimes (A_2|_E))$$

and the degree of this polynomial is less than $n-3$. Hence $C_1 + C_2 + C_3 + C_4 + C_5 + C_6 = 0$. Therefore $C_6 = 0$ because $C_1 = C_2 + C_4 = C_3 + C_5 = 0$.

By above we see that the coefficient of $t_1 \cdots t_{n-1}$ in $f(t_1,\ldots,t_{n-1})$ is zero. Therefore by the definition of intersection numbers (see [23]) we have $(A_1|_E)(A_2|_E)(\pi|_E)^*(L_1|_Z) \cdots (\pi|_E)^*(L_{n-3}|_Z) = 0$. Hence we get the assertion. $\square$

Lemma 2.4 *Let X be a smooth projective variety of dimension $n \geq 4$ and let V be a normal Gorenstein projective variety of dimension $n \geq 4$ with only terminal singularities and $\dim \mathrm{Sing}(V) \leq n-4$. Let $\pi : X \to V$ be a birational morphism such that $X \backslash \pi^{-1}(\mathrm{Sing}(V)) \cong V \backslash \mathrm{Sing}(V)$. Let E_π be the π-exceptional divisor on X with $K_X = \pi^*(K_V) + E_\pi$ and $L_1,\ldots,L_{n-3}$ line bundles on V. Then $c_2(X) E_\pi (\pi^*(L_1)) \cdots (\pi^*(L_{n-3})) = 0$.*

Proof. Let $A_1,\ldots,A_{n-3}$ and A be line bundles on X. Then by [16, Theorem 2.4] we get the following.

(1)
$$\chi_3^H(X, A_1,\ldots,A_{n-3}; A)$$
$$= \sum_{k=0}^3 \left(\sum_{(t_1,\cdots,t_{n-3}) \in S(n-3)_{n-k}^+} \frac{(-1)^k}{(t_1)! \cdots (t_{n-3})!} A_1^{t_1} \cdots A_{n-3}^{t_{n-3}} \right) R_k(X, A).$$

Here we note that

$$
\begin{aligned}
R_0(X, A) &= 1, \\
R_1(X, A) &= T_1(X) + \operatorname{ch}(A)_1 = \frac{1}{2}c_1(X) + A, \\
R_2(X, A) &= T_2(X) + \operatorname{ch}(A)_1 T_1(X) + \operatorname{ch}(A)_2 \\
&= \frac{1}{12}(c_2(X) + c_1(X)^2) + \frac{1}{2}c_1(X)A + \frac{1}{2}A^2, \\
R_3(X, A) &= T_3(X) + \operatorname{ch}(A)_1 T_2(X) + \operatorname{ch}(A)_2 T_1(X) + \operatorname{ch}(A)_3 \\
&= \frac{1}{24}c_1(X)c_2(X) + \frac{1}{12}(c_2(X) + c_1(X)^2)A + \frac{1}{4}c_1(X)A^2 + \frac{1}{6}A^3.
\end{aligned}
$$

Here we put $A_i = \pi^*(L_i)$ for $i = 1, \ldots, n-3$. Then by using Lemma 2.3 and the equation (1) above we have

$$
\begin{aligned}
(2) \qquad &\chi_3^H(X, \pi^*(L_1), \ldots, \pi^*(L_{n-3}); \mathcal{O}(E_\pi)) - \chi_3^H(X, \pi^*(L_1), \ldots, \pi^*(L_{n-3}); \mathcal{O}_X) \\
&= \frac{1}{12}c_2(X)E_\pi \pi^*(L_1) \cdots \pi^*(L_{n-3}).
\end{aligned}
$$

By Grauert-Riemenschneider's theorem we see that for every $i \geq 1$

$$
\begin{aligned}
0 &= R^i \pi_*(\mathcal{O}(K_X)) \\
&= R^i \pi_*(\pi^*(K_V) + E_\pi) \\
&= R^i \pi_*(\mathcal{O}(E_\pi)) \otimes K_V.
\end{aligned}
$$

Hence $R^i \pi_*(\mathcal{O}(E_\pi)) = 0$ for every $i \geq 1$. We also note that $R^i \pi_*(\mathcal{O}_X) = 0$ for every $i \geq 1$ because V has only rational singularities. So we see that for every integer i with $i \geq 0$ we have

$$
\begin{aligned}
&h^i(\pi^*(L_1)^{\otimes t_1} \otimes \cdots \otimes \pi^*(L_{n-3})^{\otimes t_{n-3}} \otimes \mathcal{O}(E_\pi)) \\
&= h^i((L_1)^{\otimes t_1} \otimes \cdots \otimes (L_{n-3})^{\otimes t_{n-3}} \otimes \pi_*(\mathcal{O}(E_\pi))) \\
&= h^i((L_1)^{\otimes t_1} \otimes \cdots \otimes (L_{n-3})^{\otimes t_{n-3}}) \\
&= h^i(\pi^*(L_1)^{\otimes t_1} \otimes \cdots \otimes \pi^*(L_{n-3})^{\otimes t_{n-3}}).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\chi(\pi^*(L_1)^{\otimes t_1} \otimes \cdots \otimes \pi^*(L_{n-3})^{\otimes t_{n-3}} \otimes \mathcal{O}(E_\pi)) \\
&= \chi(\pi^*(L_1)^{\otimes t_1} \otimes \cdots \otimes \pi^*(L_{n-3})^{\otimes t_{n-3}}).
\end{aligned}
$$

In particular, we have

$$
\begin{aligned}
(3) \qquad &\chi_3^H(X, \pi^*(L_1), \ldots, \pi^*(L_{n-3}); \mathcal{O}(E_\pi)) \\
&= \chi_3^H(X, \pi^*(L_1), \ldots, \pi^*(L_{n-3}); \mathcal{O}_X).
\end{aligned}
$$

So by (2) and (3) we get the assertion. $\qquad\qquad\square$

3   The case where $K_X + L$ is nef

In this section, we assume that (X, L) satisfies the following assumption (SRE).

(SRE): (X, L) is a polarized variety of dimension n such that X is a normal Gorenstein projective variety with only isolated terminal singularities.

Here we note that this condition appears when we take the second reduction of polarized manifolds of even dimension.

First we prove the following proposition.

Proposition 3.1 $\mathcal{M}_n^{NEF}(\text{SRE}) \neq \emptyset$

Proof. Let $(X, L) \in \mathcal{P}_N^{NEF}(\text{SRE})$. Since $(m-1)K_X + mL$ is nef and big for every integer $m \geq 1$, we have $h^i(m(K_X + L)) = 0$ for every integer $i \geq 1$ by the Kawamata-Viehweg vanishing theorem. Hence $h^0(t(K_X + L)) = \chi(t(K_X + L))$ for every integer $t \geq 1$. Since $\chi(t(K_X + L))$ is a polynomial in t of degree at most n, there exists an integer p such that $1 \leq p \leq n+1$ and $h^0(p(K_X + L)) > 0$. Using Lemma 2.1, we have $h^0((n+1)!(K_X + L)) > 0$ for any $(X, L) \in \mathcal{P}_N^{NEF}(\text{SRE})$. Therefore $(n+1)! \in \mathcal{M}_n^{NEF}(\text{SRE})$ and we get the assertion. $\square$

Next we will prove the following theorem.

Theorem 3.1 *Let (X, L) be a polarized variety of dimension $n \geq 4$ which satisfies the assumption* (SRE)*, and let Y be a normal projective variety of dimension 3. Assume that there exists a fiber space $f : X \to Y$ such that $K_X + L = f^*(H)$ for some ample line bundle H on Y. Then the following hold:*

(1) *If $h^1(\mathcal{O}_X) \geq 1$, then $h^0(m(K_X + L)) \geq 1$ for every positive integer m.*

(2) *If $h^1(\mathcal{O}_X) = 0$, then $h^0(m(K_X + L)) \geq 1$ for every integer m with $m \geq 2$.*

Proof. Let $\delta : T \to Y$ be a resolution of Y such that $T \setminus \delta^{-1}(\text{Sing}(Y)) \cong Y \setminus \text{Sing}(Y)$. Then there exist a smooth projective variety X', a birational morphism $\mu : X' \to X$ and a fiber space $f' : X' \to T$ such that $f \circ \mu = \delta \circ f'$.
(1) The case where $h^1(\mathcal{O}_X) > 0$.
(1.1) First we note that $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_Y) \leq h^1(\mathcal{O}_T) \leq h^1(\mathcal{O}_{X'}) = h^1(\mathcal{O}_X)$. Hence $h^1(\mathcal{O}_Y) = h^1(\mathcal{O}_T)$, and by [27, Lemma 0.3.3] or [4, Lemma 2.4.1 and Remark 2.4.2], we see that Y has the Albanese map. Let $\alpha : Y \to \text{Alb}(Y)$ be the Albanese map of Y and let $h := \alpha \circ f \circ \mu$.
(a) First we consider the case where $\dim h(X') = 3$. By [20, Corollary 10.7 in Chapter III] any general fiber F_h of h can be written as follows: $F_h = \cup_{i=1}^r F_i$, where F_i is a smooth projective variety of dimension $n - 3$. We note that F_i is a fiber of $f \circ \mu$ and we can take μ such that $K_{X'}|_{F_h} = \mu^*(K_X)|_{F_h}$ because $\dim \text{Sing}(X) \leq 0$ and $\dim \text{Sing}(Y) \leq 1$. Hence

$$
\begin{aligned}
h^0((K_{X'} + \mu^*(L))|_{F_h}) &= \sum_{i=1}^r h^0(\mu^*(f^*(H))|_{F_i}) \\
&= \sum_{i=1}^r h^0(\mathcal{O}_{F_i}) \\
&> 0.
\end{aligned}
$$

By [8, Lemma 4.1] we have $h^0(K_X + L) = h^0(K_{X'} + \mu^*(L)) > 0$.
(b) Next we consider the case of $\dim h(X') = 2$. Then $\dim \alpha(Y) = 2$ and let $Y \to Z \to \alpha(Y)$ be the Stein factorization of α. We set $\alpha_1 : Y \to Z$, $\delta_1 := \alpha_1 \circ \delta$ and $h_1 := \delta_1 \circ f'$. Then we note that h_1 has connected fibers. Let F_{h_1} (resp. F_{δ_1}) be a general fiber of h_1 (resp. δ_1). As in the case (a) above, we can take μ such that $K_{X'}|_{F_{h_1}} = \mu^*(K_X)|_{F_{h_1}}$ because $\dim \text{Sing}(X) \leq 0$ and $\dim \text{Sing}(Y) \leq 1$. Then F_{h_1} and F_{δ_1} are smooth with $\dim F_{h_1} = n - 2$ and $\dim F_{\delta_1} = 1$, $f|_{F_{h_1}} : F_{h_1} \to F_{\delta_1}$ is a fiber space such that $K_{F_{h_1}} + (\mu^*L)|_{F_{h_1}} = (\mu^* \circ f^*(H))|_{F_{h_1}} = (f'|_{F_{h_1}})^*(\delta^*(H)|_{F_{\delta_1}})$. Here we note that $\delta^*(H)|_{F_{\delta_1}}$ is ample because $\dim F_{\delta_1} = 1$ and $\deg \delta^*(H)|_{F_{\delta_1}} > 0$. By [18, Theorem 4.1], we have $h^0(K_{F_{h_1}} + \mu^*(L)|_{F_{h_1}}) > 0$. Therefore by [8, Lemma 4.1] we get $h^0(K_X + L) = h^0(K_{X'} + \mu^*(L)) > 0$.
(c) Next we consider the case of $\dim h(X') = 1$. Then $\alpha(Y)$ is a smooth curve and $\alpha : Y \to \alpha(Y)$ has connected fibers (see [4, Lemma 2.4.5]). Let F_h (resp. F_α) be a general fiber of h (resp. α). Then F_h is smooth and F_α is a projective variety with $\dim F_\alpha = 2$ and $(f \circ \mu)|_{F_h} : F_h \to F_\alpha$ is a surjective morphism with connected fibers. By taking its Stein factorization, if necessary, we may assume that F_α is normal. Since $K_{F_h} + L_{F_h} = \mu^*(f^*(H))|_{F_h} = ((f \circ \mu)|_{F_h})^*(H|_{F_\alpha})$ and $H|_{F_\alpha}$ is

ample, by [18, Theorem 4.3] we see that $h^0(K_{F_h} + L_{F_h}) > 0$. Therefore by [8, Lemma 4.1] we get $h^0(K_X + L) = h^0(K_{X'} + \mu^*(L)) > 0$.

From (a), (b) and (c) above, we get $h^0(K_X + L) > 0$. Therefore we see that $h^0(m(K_X + L)) > 0$ by Lemma 2.1.

(2) The case where $h^1(\mathcal{O}_X) = 0$.

Then $h^1(\mathcal{O}_Y) = 0$. If $h^0(K_X + L) > 0$, then we get the assertion by Lemma 2.1. So we assume that $h^0(K_X + L) = 0$.

Since $R^i f_*(p(K_X + L)) = R^i f_*(K_X + ((p-1)K_X + pL)) = 0$ for every integers i and p with $i > 0$ and $p > 0$, we have $h^i(p(K_K + L)) = h^i(f_*(p(K_X + L))) = h^i(pH)$ for every integer i and p with $i \geq 0$ and $p > 0$. Therefore $\chi(pH) = h^0(pH)$ for every positive integer p. Since $h^0(K_X + L) = 0$, we get $\chi(H) = 0$. Let t be an indeterminate. Then $\chi(tH)$ is a polynomial of t whose degree is 3. Because $\chi(H) = 0$, we can write $\chi(tH) = d(t-1)(t^2 + at + b)$, where a and d are real numbers. On the other hand, we set

$$\chi(tH) = \sum_{j=0}^{3} \chi_j(Y, H) \binom{t+j-1}{j}$$

Then $\chi_3(Y, H) = 6d$, $\chi_2(Y, H) + \chi_3(Y, H) = 2d(a-1)$, $2\chi_3(Y, H) + 3\chi_2(Y, H) + 6\chi_1(Y, H) = 6d(b-a)$ and $\chi_0(Y, H) = -bd$. Since $H^3 = \chi_3(Y, H) = 6d$, we have $g_1(Y, H) = 1 - \chi_2(Y, H) = 1 - 2d(a-4)$ and $g_2(Y, H) = -1 + h^1(\mathcal{O}_Y) + \chi_1(Y, H) = d(b - 2a + 2) - 1$.

Next we prove the following claim.

Claim 3.1 $a \geq -1/2$.

Proof. Let $\delta : T \to Y$ be a resolution of Y such that $T \setminus \delta^{-1}(\mathrm{Sing}(Y)) \cong Y \setminus \mathrm{Sing}(Y)$. Then there exist a smooth projective variety X', a birational morphism $\mu : X' \to X$ and a fiber space $f' : X' \to T$ such that $f \circ \mu = \delta \circ f'$. Let $L' = \mu^*(L)$. By the same argument as in the proof of [18, Theorem 4.3], we have $0 \leq (\delta^*(H))^3 - K_T(\delta^*(H))^2$ because $K_{X'/T} + L'$ is pseudo-effective. On the other hand,

$$(\delta^*(H))^3 - K_T(\delta^*(H))^2$$
$$= 3\chi_0^H(T, \delta^*(H)) + 2\chi_1^H(T, \delta^*(H))$$
$$= 3\chi_0^H(Y, H) + 2\chi_1^H(Y, H)$$
$$= 2d(2a+1).$$

Therefore $2d(1 + 2a) \geq 0$ and $a \geq -1/2$ since $d > 0$. $\qquad\square$

Assume that $h^0(2(K_X + L)) = 0$. Then $b = -2a - 4$ because $\chi(2H) = 0$. Hence by Claim 3.1 we have $g_2(Y, H) = d(b - 2a + 2) - 1 = d(-4a - 2) - 1 \leq -1$. By [19, Lemma 3.1], we see that $\chi_2^H(T, \delta^*(H)) \leq \chi_2^H(Y, H)$ holds. We also have $h^1(\mathcal{O}_T) = h^1(\mathcal{O}_Y) = 0$ since $h^1(\mathcal{O}_X) = 0$ and X has only rational singularities. Therefore we get $g_2(T, \delta^*(H)) \leq g_2(Y, H)$.

If $\kappa(T) = -\infty$, then we see that $g_2(T, \delta^*(H)) \geq 0$ by [19, Proposition 3.1].

If $\kappa(T) \geq 0$, then by [9, (4.2) Theorem], there exists a quasi-polarized variety (V_1, H_1) of dimension 3 such that V_1 is a normal projective variety with only $\mathbb{Q}$-factorial terminal singularities, $(T, \delta^*(H))$ and (V_1, H_1) are birationally equivalent and $K_{V_1} + 2H_1$ is nef. Let $\pi : X_1 \to V_1$ be a resolution of V_1. Then by [19, Theorem 4.3] we have

$$g_2(T, \delta^*(H))$$
$$\geq -1 + \frac{1}{12}\pi^*(K_{V_1})(\pi^*(K_{V_1} + 2H_1))\pi^*(H_1) - \frac{1}{36}\pi^*((K_{V_1} + 2H_1))\pi^*(H_1)^2 + \frac{1}{9}(\pi^*(H_1))^3.$$

We also note that there exist a smooth projective variety X_2 of dimension 3 and birational morphisms $\pi_2 : X_2 \to X_1$ and $\beta : X_2 \to T$ such that $\beta^*(\delta^*(H)) = \pi_2^*(\pi^*(H_1))$. Let $\gamma := \pi \circ \pi_2$.

Then $K_{X_2} = \beta^*(K_T) + E_\beta$ and $K_{X_2} = \gamma^*(K_{V_1}) + E_\gamma$ hold, where E_β (resp. E_γ) is a β-exceptional (resp. γ-exceptional) effective $\mathbb{Q}$-Cartier divisor. Since we assume $\kappa(T) \geq 0$, we have
$0 \leq K_{X_2}\gamma^*(K_{V_1} + 2H_1)\gamma^*(H_1) = \gamma^*(K_{V_1})\gamma^*(K_{V_1} + 2H_1)\gamma^*(H_1) = \pi^*(K_{V_1})\pi^*(K_{V_1} + 2H_1)\pi^*(H_1)$.

Moreover since

$$
\begin{aligned}
\pi^*(K_{V_1} + 2H_1)\pi^*(H_1)^2 &= \gamma^*(K_{V_1} + 2H_1)\gamma^*(H_1)^2 \\
&= (K_{X_2} + 2\gamma^*(H_1))\gamma^*(H_1)^2 \\
&= (\beta^*(K_T) + E_\beta + 2\beta^*\delta^*(H))\beta^*\delta^*(H)^2 \\
&= (K_T + 2\delta^*(H))\delta^*(H)^2
\end{aligned}
$$

and $(\pi^*(H_1))^3 = (\delta^*(H))^3$, we have

$$
\begin{aligned}
&-\frac{1}{36}\pi^*(K_{V_1} + 2H_1)\pi^*(H_1)^2 + \frac{1}{9}(\pi^*(H_1))^3 \\
&= -\frac{1}{36}K_T\delta^*(H)^2 + \frac{1}{18}(\delta^*(H))^3 \\
&= \frac{1}{36}(4\chi_0^H(T, \delta^*(H)) + 2\chi_1^H(T, \delta^*(H))) \\
&= \frac{1}{36}(4\chi_0^H(Y, H) + 2\chi_1^H(Y, H)) \\
&= \frac{1}{18}(2\chi_0^H(Y, H) + \chi_1^H(Y, H)) \\
&= \frac{1}{18}(12d + 2d(a - 1) - 6d) \\
&= \frac{1}{9}(2d + ad).
\end{aligned}
$$

Since $a \geq -1/2$ by Claim 3.1, we have

$$
-\frac{1}{36}\pi^*((K_{V_1} + 2H_1))\pi^*(H_1)^2 + \frac{1}{9}(\pi^*(H_1))^3 \geq \frac{1}{9}(2d - \frac{1}{2}d) = \frac{1}{6}d.
$$

Therefore

$$
g_2(Y, H) \geq \frac{1}{6}d - 1.
$$

On the other hand, as we said before, $g_2(Y, H) \leq d(-4a - 2) - 1 \leq -1$ holds by Claim 3.1. But this is impossible because

$$
d = \frac{1}{6}\chi_0^H(Y, H) = \frac{1}{6}H^3 > 0.
$$

Hence $h^0(2(K_X + L)) \neq 0$.

Assume that $h^0(3(K_X + L)) = 0$. Then $b = -3a - 9$ because $\chi(3H) = 0$. Hence $g_2(Y, H) = d(b - 2a + 2) - 1 = d(-5a - 7) - 1 < 0$ by Claim 3.1. But this is impossible by the same argument as above. Hence $h^0(3(K_X + L)) \neq 0$.

Therefore $h^0(m(K_X + L)) > 0$ for every integer m with $m \geq 2$ by Lemmas 2.1 and 2.2. Hence we get the assertion. $\qquad\square$

Next we consider the case where X is a normal Gorenstein projective variety of dimension 4 with only isolated terminal singularities.

Theorem 3.2 *Let (X, L) be a polarized variety of dimension 4 which satisfies the assumption* (SRE). *Assume that $K_X + L$ is nef.*

(1) *If $0 \leq \kappa(K_X + L) \leq 2$, then $h^0(m(K_X + L)) > 0$ for every integer m with $m \geq 1$.*

(2) *If* $\kappa(K_X + L) = 3$, *then* $h^0(m(K_X + L)) > 0$ *for every integer* m *with* $m \geq 2$.

Proof. (i) If $\kappa(K_X + L) = 0$, then we can prove that $\mathcal{O}(K_X + L) = \mathcal{O}_X$ by [4, Lemma 3.3.2]. Then $h^0(K_X + L) = 1$.

(ii) If $\kappa(K_X + L) = 1$ (resp. 2), then there exist a normal projective variety Y with $\dim Y = 1$ (resp. 2) and a fiber space $f : X \to Y$ such that $K_X + L = f^*(H)$ for some ample line bundle H on Y. By the same argument as in the proof of [18, Theorem 4.1] (resp. [18, Theorem 4.3]) we can prove $h^0(m(K_X + L)) > 0$ for any $m \geq 1$.

(iii) If $\kappa(K_X + L) = 3$, then there exist a normal projective variety Y with $\dim Y = 3$ and a fiber space $f : X \to Y$ such that $K_X + L = f^*(H)$ for some ample line bundle H on Y. By Theorem 3.1 we get $h^0(m(K_X + L)) > 0$ for any $m \geq 2$. $\qquad\square$

Theorem 3.3 *Let* (X, L) *be a polarized variety of dimension* 4 *which satisfies the assumption* (SRE). *Assume that* $K_X + L$ *is nef and big. Then* $h^0(m(K_X + L)) > 0$ *for every integer* m *with* $m \geq 4$.

Proof. (i) First we consider the case of $m = 4$.

Claim 3.2 $h^0(4(K_X + L)) > 0$.

Proof. Assume that $h^0(4(K_X + L)) = 0$. Then by Lemma 2.1, we get $h^0(2(K_X + L)) = 0$. Therefore we have

$$
\begin{aligned}
0 &\geq h^0(2(K_X + L)) - h^0(K_X + L), \\
0 &\leq h^0(3(K_X + L)) - h^0(2(K_X + L)), \\
0 &\geq h^0(4(K_X + L)) - h^0(3(K_X + L)).
\end{aligned}
$$

On the other hand by Theorem 2.1

$$
\begin{aligned}
(4) \qquad & h^0(m(K_X + L)) - h^0((m-1)(K_X + L)) \\
& = g_3(X, K_X + L) + g_2(X, K_X + L, (m-2)K_X + (m-1)L) - h^2(\mathcal{O}_X).
\end{aligned}
$$

By using the above,

$$
\begin{aligned}
(5) \qquad 0 &\geq g_3(X, K_X + L) + g_2(X, L, K_X + L) - h^2(\mathcal{O}_X), \\
(6) \qquad 0 &\leq g_3(X, K_X + L) + g_2(X, K_X + L, K_X + 2L) - h^2(\mathcal{O}_X), \\
(7) \qquad 0 &\geq g_3(X, K_X + L) + g_2(X, K_X + L, 2K_X + 3L) - h^2(\mathcal{O}_X).
\end{aligned}
$$

By (5) and (6) we get

$$
(8) \qquad g_2(X, K_X + L, K_X + 2L) \geq g_2(X, L, K_X + L).
$$

By (6) and (7) we get

$$
(9) \qquad g_2(X, K_X + L, K_X + 2L) \geq g_2(X, K_X + L, 2K_X + 3L).
$$

On the other hand by Proposition 2.1

$$
\begin{aligned}
(10) \qquad g_2(X, K_X + L, K_X + 2L) &= g_2(X, K_X + L, K_X + L) + g_2(X, K_X + L, L) \\
& \quad + g_1(X, K_X + L, K_X + L, L) - h^1(\mathcal{O}_X)
\end{aligned}
$$

and

$$
\begin{aligned}
(11) \quad g_2(X, K_X + L, 2K_X + 3L) &= g_2(X, K_X + L, K_X + L) + g_2(X, K_X + L, K_X + 2L) \\
& \quad + g_1(X, K_X + L, K_X + L, K_X + 2L) - h^1(\mathcal{O}_X).
\end{aligned}
$$

By (8) and (10), we have

(12) $$g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, L) - h^1(\mathcal{O}_X) \geq 0.$$

By (9) and (11), we have

(13) $$g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, K_X + 2L) - h^1(\mathcal{O}_X) \leq 0.$$

Hence by (12) and (13) we get

$$g_1(X, K_X + L, K_X + L, L) - g_1(X, K_X + L, K_X + L, K_X + 2L) \geq 0.$$

On the other hand since $K_X + L$ is nef and 1-big we have

$$g_1(X, K_X + L, K_X + L, L) - g_1(X, K_X + L, K_X + L, K_X + 2L)$$
$$= -\frac{1}{2}(K_X + L)^3(4K_X + 5L)$$
$$< 0.$$

This is a contradiction. Therefore $h^0(4(K_X + L)) > 0$. $\qquad\square$

(ii) Next we are going to prove the following:

Claim 3.3 $h^0(3(K_X + L)) \neq 0$ *or* $h^0(5(K_X + L)) \neq 0$.

Proof. Assume that $h^0(3(K_X + L)) = 0$ and $h^0(5(K_X + L)) = 0$. Then the following hold:

$$\begin{aligned}
0 &\geq h^0(3(K_X + L)) - h^0(2(K_X + L)), \\
0 &\leq h^0(4(K_X + L)) - h^0(3(K_X + L)), \\
0 &\geq h^0(5(K_X + L)) - h^0(4(K_X + L)).
\end{aligned}$$

By (4), we have

$$\begin{aligned}
0 &\geq g_3(X, K_X + L) + g_2(X, K_X + L, K_X + 2L) - h^2(\mathcal{O}_X), \\
0 &\leq g_3(X, K_X + L) + g_2(X, K_X + L, 2K_X + 3L) - h^2(\mathcal{O}_X), \\
0 &\geq g_3(X, K_X + L) + g_2(X, K_X + L, 3K_X + 4L) - h^2(\mathcal{O}_X).
\end{aligned}$$

Therefore we get

(14) $$g_2(X, K_X + L, 2K_X + 3L) \geq g_2(X, K_X + L, K_X + 2L),$$
(15) $$g_2(X, K_X + L, 2K_X + 3L) \geq g_2(X, K_X + L, 3K_X + 4L).$$

On the other hand by Proposition 2.1 we have

$$g_2(X, K_X + L, 2K_X + 3L)$$
$$= g_2(X, K_X + L, K_X + 2L) + g_2(X, K_X + L, K_X + L)$$
$$+ g_1(X, K_X + L, K_X + L, K_X + 2L) - h^1(\mathcal{O}_X)$$

and

$$g_2(X, K_X + L, 3K_X + 4L)$$
$$= g_2(X, K_X + L, 2K_X + 3L) + g_2(X, K_X + L, K_X + L)$$
$$+ g_1(X, K_X + L, K_X + L, 2K_X + 3L) - h^1(\mathcal{O}_X).$$

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Hence

$$g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, K_X + 2L) - h^1(\mathcal{O}_X) \geq 0,$$
$$g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, 2K_X + 3L) - h^1(\mathcal{O}_X) \leq 0,$$

and therefore we get

$$g_1(X, K_X + L, K_X + L, K_X + 2L) - g_1(X, K_X + L, K_X + L, 2K_X + 3L) \geq 0.$$

But since $K_X + L$ is nef and 1-big and $3K_X + (7/2)L$ is ample, we have

$$g_1(X, K_X + L, K_X + L, K_X + 2L) - g_1(X, K_X + L, K_X + L, 2K_X + 3L)$$
$$= -(K_X + L)^3 \left(3K_X + \frac{7}{2}L \right)$$
$$\leq 0.$$

This is a contradiction. This completes the proof of this claim. $\qquad\square$

(ii.1) Next we consider the case of $h^0(3(K_X + L)) > 0$.
If $h^0(3(K_X + L)) > 0$, then by using the positivity of $h^0(4(K_X + L))$, we have $h^0(m(K_X + L)) > 0$ for every integer m with $m \geq 6$ by Lemmas 2.1 and 2.2.

Claim 3.4 *If* $h^0(3(K_X + L)) > 0$, *then* $h^0(5(K_X + L)) > 0$.

Proof. Assume that $h^0(5(K_X + L)) = 0$. If $h^0(2(K_X + L)) > 0$, then by Lemma 2.1 we see that $h^0(5(K_X + L)) > 0$. So we may assume that $h^0(2(K_X + L)) = 0$. Then

$$0 \geq h^0(2(K_X + L)) - h^0(K_X + L),$$
$$0 \leq h^0(3(K_X + L)) - h^0(2(K_X + L)),$$
$$0 \geq h^0(5(K_X + L)) - h^0(4(K_X + L)).$$

By (4), we have

$$0 \geq g_3(X, K_X + L) + g_2(X, K_X + L, L) - h^2(\mathcal{O}_X),$$
$$0 \leq g_3(X, K_X + L) + g_2(X, K_X + L, K_X + 2L) - h^2(\mathcal{O}_X),$$
$$0 \geq g_3(X, K_X + L) + g_2(X, K_X + L, 3K_X + 4L) - h^2(\mathcal{O}_X).$$

On the other hand by Proposition 2.1 we see that

$$g_2(X, K_X + L, K_X + 2L)$$
$$= g_2(X, K_X + L, L) + g_2(X, K_X + L, K_X + L)$$
$$\quad + g_1(X, K_X + L, K_X + L, L) - h^1(\mathcal{O}_X)$$

and

$$g_2(X, K_X + L, 3K_X + 4L)$$
$$= g_2(X, K_X + L, K_X + 2L) + g_2(X, K_X + L, 2K_X + 2L)$$
$$\quad + g_1(X, K_X + L, 2(K_X + L), K_X + 2L) - h^1(\mathcal{O}_X)$$
$$= g_2(X, K_X + L, K_X + 2L) + 2g_2(X, K_X + L, K_X + L)$$
$$\quad + g_1(X, K_X + L, K_X + L, K_X + L) + g_1(X, K_X + L, 2(K_X + L), K_X + 2L)$$
$$\quad - 2h^1(\mathcal{O}_X).$$

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Hence

$$
\begin{aligned}
0 \;\leq\; & g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, L) - h^1(\mathcal{O}_X), \\
0 \;\geq\; & 2g_2(X, K_X + L, K_X + L) - 2h^1(\mathcal{O}_X) \\
& + g_1(X, K_X + L, K_X + L, K_X + L) + g_1(X, K_X + L, 2(K_X + L), K_X + 2L)
\end{aligned}
$$

and therefore we get

$$
\begin{aligned}
& 2g_1(X, K_X + L, K_X + L, L) \\
& \geq g_1(X, K_X + L, K_X + L, K_X + L) + g_1(X, K_X + L, 2(K_X + L), K_X + 2L).
\end{aligned}
$$

But since $K_X + L$ is nef and 1-big and $7K_X + (17/2)L$ is ample, we have

$$
\begin{aligned}
& 2g_1(X, K_X + L, K_X + L, L) - g_1(X, K_X + L, K_X + L, K_X + L) \\
& - g_1(X, K_X + L, 2(K_X + L), K_X + 2L) \\
& = -(K_X + L)^3 \left(7K_X + \frac{17}{2}L \right) \\
& \leq 0.
\end{aligned}
$$

This is a contradiction. This completes the proof of this claim. $\qquad\square$

(ii.2) Next we consider the case of $h^0(5(K_X + L)) > 0$.

Claim 3.5 *If $h^0(5(K_X + L)) > 0$, then $h^0(m(K_X + L)) > 0$ for every integer m with $m \geq 5$.*

Proof. If $h^0(5(K_X + L)) > 0$, then by using the positivity of $h^0(4(K_X + L))$, we have $h^0(m(K_X + L)) > 0$ for $m = 5, 8, 9, 10$ and $m \geq 12$ by Lemmas 2.1 and 2.2.

So we consider the case where $m = 6$ (resp. 7, 11).

Assume that $h^0(6(K_X + L)) = 0$ (resp. $h^0(7(K_X + L)) = 0$, $h^0(11(K_X + L)) = 0$). If $h^0(3(K_X + L)) > 0$, then by Lemma 2.1 (resp. Lemma 2.1 and Claim 3.2, Lemma 2.1 and Claim 3.2) we see that $h^0(6(K_X + L)) > 0$ (resp. $h^0(7(K_X + L)) > 0$, $h^0(11(K_X + L)) > 0$). So we may assume that $h^0(3(K_X + L)) = 0$. Then

$$
\begin{aligned}
0 \;\geq\; & h^0(3(K_X + L)) - h^0(2(K_X + L)), \\
0 \;\leq\; & h^0(4(K_X + L)) - h^0(3(K_X + L)), \\
0 \;\geq\; & h^0(6(K_X + L)) - h^0(5(K_X + L)) \\
(\text{resp. } 0 \;\geq\; & h^0(7(K_X + L)) - h^0(6(K_X + L)), \\
0 \;\geq\; & h^0(11(K_X + L)) - h^0(10(K_X + L))).
\end{aligned}
$$

By (4), we have

$$
\begin{aligned}
0 \;\geq\; & g_3(X, K_X + L) + g_2(X, K_X + L, K_X + 2L) - h^2(\mathcal{O}_X), \\
0 \;\leq\; & g_3(X, K_X + L) + g_2(X, K_X + L, 2K_X + 3L) - h^2(\mathcal{O}_X), \\
0 \;\geq\; & g_3(X, K_X + L) + g_2(X, K_X + L, 4K_X + 5L) - h^2(\mathcal{O}_X) \\
(\text{resp. } 0 \;\geq\; & g_3(X, K_X + L) + g_2(X, K_X + L, 5K_X + 6L) - h^2(\mathcal{O}_X), \\
0 \;\geq\; & g_3(X, K_X + L) + g_2(X, K_X + L, 9K_X + 10L) - h^2(\mathcal{O}_X)).
\end{aligned}
$$

Hence

$$
\begin{aligned}
g_2(X, K_X + L, 2K_X + 3L) \;\geq\; & g_2(X, K_X + L, K_X + 2L), \\
g_2(X, K_X + L, 2K_X + 3L) \;\geq\; & g_2(X, K_X + L, 4K_X + 5L) \\
(\text{resp. } g_2(X, K_X + L, 2K_X + 3L) \;\geq\; & g_2(X, K_X + L, 5K_X + 6L) \\
g_2(X, K_X + L, 2K_X + 3L) \;\geq\; & g_2(X, K_X + L, 9K_X + 10L)).
\end{aligned}
$$

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On the other hand by Proposition 2.1 we have

$$g_2(X, K_X + L, 2K_X + 3L)$$
$$= g_2(X, K_X + L, K_X + 2L) + g_2(X, K_X + L, K_X + L)$$
$$+ g_1(X, K_X + L, K_X + L, K_X + 2L) - h^1(\mathcal{O}_X),$$

$$g_2(X, K_X + L, 4K_X + 5L) = g_2(X, K_X + L, 2K_X + 3L) + 2g_2(X, K_X + L, K_X + L)$$
$$+ g_1(X, K_X + L, K_X + L, K_X + L)$$
$$+ g_1(X, K_X + L, 2K_X + 3L, 2K_X + 2L) - 2h^1(\mathcal{O}_X),$$

$$(resp. g_2(X, K_X + L, 5K_X + 6L) = g_2(X, K_X + L, 2K_X + 3L) + 3g_2(X, K_X + L, K_X + L)$$
$$+ g_1(X, K_X + L, K_X + L, K_X + L)$$
$$+ g_1(X, K_X + L, K_X + L, 2K_X + 2L)$$
$$+ g_1(X, K_X + L, 2K_X + 3L, 3K_X + 3L) - 3h^1(\mathcal{O}_X)$$

and

$$g_2(X, K_X + L, 9K_X + 10L) = g_2(X, K_X + L, 2K_X + 3L) + 7g_2(X, K_X + L, K_X + L)$$
$$+ \sum_{k=1}^{6} g_1(X, K_X + L, K_X + L, k(K_X + L))$$
$$+ g_1(X, K_X + L, 2K_X + 3L, 7K_X + 7L) - 7h^1(\mathcal{O}_X)).$$

Hence we have

$$g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, K_X + 2L) - h^1(\mathcal{O}_X) \geq 0,$$

$$2g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, K_X + L)$$
$$+ g_1(X, K_X + L, 2K_X + 3L, 2K_X + 2L) - 2h^1(\mathcal{O}_X) \leq 0$$

$$(resp. \quad 3g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, K_X + L)$$
$$+ g_1(X, K_X + L, K_X + L, 2K_X + 2L)$$
$$+ g_1(X, K_X + L, 2K_X + 3L, 3K_X + 3L) - 3h^1(\mathcal{O}_X) \geq 0,$$

$$7g_2(X, K_X + L, K_X + L) + \sum_{k=1}^{6} g_1(X, K_X + L, K_X + L, k(K_X + L))$$
$$+ g_1(X, K_X + L, 2K_X + 3L, 7K_X + 7L) - 7h^1(\mathcal{O}_X) \leq 0).$$

Therefore

$$2g_1(X, K_X + L, K_X + L, K_X + 2L) \geq g_1(X, K_X + L, K_X + L, K_X + L)$$
$$+ g_1(X, K_X + L, 2K_X + 3L, 2K_X + 2L)$$

$$(resp. \ 3g_1(X, K_X + L, K_X + L, K_X + 2L) \geq g_1(X, K_X + L, 2K_X + 3L, 3K_X + 3L)$$
$$+ \sum_{k=1}^{2} g_1(X, K_X + L, K_X + L, k(K_X + L)),$$

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$$7g_1(X, K_X + L, K_X + L, K_X + 2L) \geq g_1(X, K_X + L, 2K_X + 3L, 7K_X + 7L)$$
$$+ \sum_{k=1}^{6} g_1(X, K_X + L, K_X + L, k(K_X + L))).$$

On the other hand, since $K_X + L$ is nef and big, we have

$$2g_1(X, K_X + L, K_X + L, K_X + 2L)$$
$$-g_1(X, K_X + L, K_X + L, K_X + L)$$
$$-g_1(X, K_X + L, 2K_X + 3L, 2K_X + 2L)$$
$$= -(K_X + L)^3 \left(10K_X + \frac{23}{2}L \right)$$
$$< 0$$

(resp.
$$3g_1(X, K_X + L, K_X + L, K_X + 2L)$$
$$-g_1(X, K_X + L, 2K_X + 3L, 3K_X + 3L)$$
$$- \sum_{k=1}^{2} g_1(X, K_X + L, K_X + L, k(K_X + L))$$
$$= -(K_X + L)^3 (22K_X + 25L)$$
$$< 0,$$

$$7g_1(X, K_X + L, K_X + L, K_X + 2L)$$
$$-g_1(X, K_X + L, 2K_X + 3L, 7K_X + 7L)$$
$$- \sum_{k=1}^{6} g_1(X, K_X + L, K_X + L, k(K_X + L))$$
$$= -(K_X + L)^3 (140K_X + 154L)$$
$$< 0).$$

This is a contradiction. Therefore we complete the proof of Claim 3.5. $\qquad\square$

Therefore we get the assertion of Theorem 3.3. $\qquad\square$

When we study the positivity of $h^0(3(K_X + L))$, we need to study the value of the second sectional geometric genus. Here we fix some notation which will be used in the following results.

Notation 3.1 Assume that (X, L) is a polarized variety of dimension 4 which satisfies the assumption (SRE). Then let $r : X_1 \to X$ be a resolution of X such that $X_1 \setminus r^{-1}(\mathrm{Sing}(X)) \cong X \setminus \mathrm{Sing}(X)$ and let $L_1 = r^*(L)$.

First we will prove the following proposition.

Proposition 3.2 *Let (X, L) be a polarized variety of dimension 4 which satisfies the assumption (SRE). We use Notation 3.1. Assume that $K_X + L$ is nef and big. Then for any nef line bundles A_1 and A_2 on X the following hold.*

(i) $c_2(X_1)r^*(A_1)r^*(A_2) \geq -\frac{1}{8}(18K_{X_1}L_1 + 27L_1^2)r^*(A_1)r^*(A_2).$

(ii) *One of the following holds.*

(ii.1) $c_2(X_1)r^*(A_1)r^*(A_2) \geq -\frac{1}{3}(6K_{X_1}L_1 + 8L_1^2)r^*(A_1)r^*(A_2)$.

(ii.2) X is rationally connected and $h^0(K_X + 2L) = h^0(K_X + L) = 0$.

Proof. (1) First we assume that $\Omega_X \left\langle \frac{3}{4}L \right\rangle$ is not generically nef. (Here $\Omega_X \left\langle \frac{3}{4}L \right\rangle$ denotes $\mathbb{Q}$-twisted sheaf. See [21, 2.3 Definition].) Then by [21, 3.1 Theorem], there exist a smooth projective variety X' of dimension 4, a smooth projective variety Y of dimension m with $m \leq 3$, a birational morphism $\mu : X' \to X$, and a surjective morphism $\varphi : X' \to Y$ such that the following holds: The general fiber F of φ is rationally connected and $h^0(D) = 0$ for any Cartier divisor D on F such that $D \sim_{\mathbb{Q}} K_F + j\mu^*(\frac{3}{4}\mu^*(L))|_F$ for any $j \in [0, 4 - m] \cap \mathbb{Q}$. (Here $\sim_{\mathbb{Q}}$ denotes the linear equivalence of $\mathbb{Q}$-divisors.)

(1.0) Assume that $\dim Y = 0$. Then X' is rationally connected and $h^0(K_{X'} + 3\mu^*(L)) = h^0(K_{X'} + 2\mu^*(L)) = h^0(K_{X'} + \mu^*(L)) = 0$. Here we note that $\chi(\mathcal{O}_{X'}) = 1$ since $h^i(\mathcal{O}_{X'}) = 0$ for any $i \geq 1$. But by [21, 4.1 Lemma] this is impossible because

$$(K_{X'} + 3\mu^*(L))\mu^*(L)^3 = (K_X + 3L)L^3 > 0.$$

(1.1) Assume that $\dim Y = 1$. Then for the general fiber F of φ we have $h^0(K_F + 2\mu^*(L)|_F) = h^0(K_F + \mu^*(L)|_F) = 0$. But $\kappa(K_F + 2\mu^*(L)|_F) \geq 0$ holds because $K_X + L$ is nef. Hence $h^0(K_F + 2\mu^*(L)|_F) > 0$ by [19, Theorem 4.6] since $\dim F = 3$. This is impossible.

(1.2) Assume that $\dim Y = 2$. Then for the general fiber F of φ we have $h^0(K_F + \mu^*(L)|_F) = 0$. On the other hand we have $\kappa(K_F + \mu^*(L)|_F) \geq 0$ because $K_X + L$ is nef. Hence $h^0(K_F + \mu^*(L)|_F) > 0$ by [19, Proposition 2.1] since $\dim F = 2$. This is also impossible.

(1.3) Assume that $\dim Y = 3$. In this case $F \cong \mathbb{P}^1$. If $\deg \mu^*(L)|_F \geq 3$, then there exists $j \in [0, 1] \cap \mathbb{Q}$ such that $K_F + j\mu^*(\frac{3}{4}\mu^*(L))|_F$ is a Cartier divisor with $\deg K_F + j\mu^*(\frac{3}{4}\mu^*(L))|_F \geq 0$. Hence $h^0(K_F + j\mu^*(\frac{3}{4}\mu^*(L))|_F) > 0$ and this is a contradiction. Therefore $\deg \mu^*(L)|_F \leq 2$. In particular

(16) $$\deg(K_F + \mu^*(L)|_F) \leq 0.$$

On the other hand we have

(17) $$\begin{aligned} K_F + \mu^*(L)_F &= (K_{X'} + \mu^*(L))_F \\ &= (\mu^*(K_X + L) + E_\mu)_F, \end{aligned}$$

where E_μ is an effective μ-exceptional divisor. Since $K_X + L$ is nef and big, we see that $\mu^*(K_X + L)$ is also nef and big. Hence $(\mu^*(K_X + L))_F$ is nef and big (see [9, (1.4) Proposition]). So we get

(18) $$\deg \mu^*(K_X + L)_F > 0.$$

Here we note that we can take a general fiber F of φ with $F \not\subset \mathrm{Supp}(E_\mu)$. Therefore

(19) $$\deg(E_\mu)_F \geq 0.$$

By (17), (18) and (19) we have $\deg(K_F + \mu^*(L)|_F) > 0$ and this contradicts to (16).

By (1.0), (1.1), (1.2) and (1.3) we conclude that $\Omega_X \left\langle \frac{3}{4}L \right\rangle$ is generically nef. Here we note that $K_X + 3L$ is nef. Hence by [21, 2.10 Lemma] we get

$$c_2\left(\Omega_X \left\langle \frac{3}{4}L \right\rangle\right) A_1 A_2 \geq 0.$$

Namely we have

$$c_2(X)A_1 A_2 \geq -\frac{1}{8}(18K_X L + 27L^2)A_1 A_2.$$

Here we note that X has only isolated singularities and $\dim X = 4$. Hence we have $c_2(X)A_1 A_2 = c_2(X_1)r^*(A_1)r^*(A_2)$. On the other hand, we have $(18K_{X_1}L_1 + 27L_1^2)r^*(A_1)r^*(A_2) = (18K_X L +$

$27L^2)A_1A_2$. Therefore we get the assertion of (i).

(2.1) First we assume that $\Omega_X\left\langle\frac{2}{3}L\right\rangle$ is generically nef. Here we note that $K_X + \frac{8}{3}L$ is nef. Hence by [21, 2.10 Lemma] we get

$$c_2\left(\Omega_X\left\langle\frac{2}{3}L\right\rangle\right)A_1A_2 \geq 0.$$

Namely we have

$$c_2(X)A_1A_2 \geq -\frac{1}{3}(6K_XL + 8L^2)A_1A_2.$$

Here we note that X has only isolated singularities and $\dim X = 4$. Hence we have $c_2(X)A_1A_2 = c_2(X_1)r^*(A_1)r^*(A_2)$. On the other hand, we have $(6K_{X_1}L_1 + 8L_1^2)r^*(A_1)r^*(A_2) = (6K_XL + 8L^2)A_1A_2$. Therefore we get (ii.1) in the statement of Proposition 3.2.

Next we assume that $\Omega_X\left\langle\frac{2}{3}L\right\rangle$ is not generically nef. Then by [21, 3.1 Theorem], there exist a smooth projective variety X' of dimension 4, a smooth projective variety Y of dimension m with $m \leq 3$, a birational morphism $\mu : X' \to X$, and a surjective morphism $\varphi : X' \to Y$ such that the following holds: The general fiber F of φ is rationally connected and $h^0(D) = 0$ for any Cartier divisor D on F such that $D \sim_{\mathbb{Q}} K_F + j\mu^*(\frac{2}{3}\mu^*(L))_F$ for any $j \in [0, 4 - m] \cap \mathbb{Q}$.

By the same argument as in the cases (1.1), (1.2) and (1.3) above, we can prove $m = 0$. Then X' is rationally connected and $h^0(K_{X'} + 2\mu^*(L)) = h^0(K_{X'} + \mu^*(L)) = 0$. Therefore we get (ii.2). $\qquad\square$

Theorem 3.4 *Let (X, L) be a polarized variety of dimension 4 which satisfies the assumption (SRE). Assume that $K_X + L$ is nef and big. Then $h^0(3(K_X + L)) > 0$.*

Proof. Assume that $h^0(3(K_X + L)) = 0$. Then by Lemma 2.1, we get $h^0(K_X + L) = 0$. Therefore

$$\begin{aligned}
0 &\leq h^0(2(K_X + L)) - h^0(K_X + L) \\
0 &\geq h^0(3(K_X + L)) - h^0(2(K_X + L)).
\end{aligned}$$

By using (4) in the proof of Claim 3.2 we have

$$(20) \qquad 0 \leq g_3(X, K_X + L) + g_2(X, L, K_X + L) - h^2(\mathcal{O}_X)$$
$$(21) \qquad 0 \geq g_3(X, K_X + L) + g_2(X, K_X + L, K_X + 2L) - h^2(\mathcal{O}_X).$$

By (20) and (21) we get

$$g_2(X, K_X + L, K_X + 2L) \leq g_2(X, L, K_X + L).$$

On the other hand by Proposition 2.1

$$\begin{aligned}
g_2(X, K_X + L, K_X + 2L) &= g_2(X, K_X + L, K_X + L) + g_2(X, K_X + L, L) \\
&\quad + g_1(X, K_X + L, K_X + L, L) - h^1(\mathcal{O}_X).
\end{aligned}$$

Hence we get

$$(22) \qquad g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, L) - h^1(\mathcal{O}_X) \leq 0.$$

(1) First we assume that (X, L) satisfies (ii.1) in Proposition 3.2. We use Notation 3.1. Then by

[14, (2.2.A)] and [19, Lemma 3.1] we have

$$g_2(X, K_X + L, K_X + L)$$
$$= g_2(X_1, r^*(K_X + L), r^*(K_X + L))$$
$$= -1 + h^1(\mathcal{O}_{X_1}) + \frac{1}{12}(K_{X_1} + 3r^*(K_X + L))(K_{X_1} + 2r^*(K_X + L))r^*(K_X + L)^2$$
$$\quad + \frac{1}{12}c_2(X_1)r^*(K_X + L)^2 + \frac{1}{24}(2K_{X_1} + 2r^*(K_X + L))r^*(K_X + L)^3$$
$$\geq -1 + h^1(\mathcal{O}_{X_1}) + \frac{1}{12}(K_{X_1} + 3r^*(K_X + L))(K_{X_1} + 2r^*(K_X + L))r^*(K_X + L)^2$$
$$\quad - \frac{1}{36}(6K_{X_1}L_1 + 8L_1^2)r^*(K_X + L)^2 + \frac{1}{24}(2K_{X_1} + 2r^*(K_X + L))r^*(K_X + L)^3$$
$$= -1 + h^1(\mathcal{O}_{X_1}) + \frac{7}{6}(K_X + L)^4 - \frac{5}{6}(K_X + L)^3 L + \frac{1}{36}(K_X + L)^2 L^2.$$

On the other hand we have

(23) $$\qquad g_1(X, K_X + L, K_X + L, L) = 1 + \frac{3}{2}(K_X + L)^3 L.$$

Hence

$$g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, L) - h^1(\mathcal{O}_X)$$
$$\geq -1 + h^1(\mathcal{O}_{X_1}) + \frac{7}{6}(K_X + L)^4 - \frac{5}{6}(K_X + L)^3 L$$
$$\quad + \frac{1}{36}(K_X + L)^2 L^2 + 1 + \frac{3}{2}(K_X + L)^3 L - h^1(\mathcal{O}_{X_1})$$
$$= \frac{7}{6}(K_X + L)^4 + \frac{2}{3}(K_X + L)^3 L + \frac{1}{36}(K_X + L)^2 L^2$$
$$> 0.$$

This contradicts (22). Therefore this case cannot occur.

(2) Next we assume that (X, L) satisfies (ii.2) in Proposition 3.2. By using Proposition 3.2 (i), [14, (2.2.A)] and [19, Lemma 3.1], we have

$$g_2(X, K_X + L, K_X + L)$$
$$= g_2(X_1, r^*(K_X + L), r^*(K_X + L))$$
$$\geq -1 + h^1(\mathcal{O}_{X_1}) + \frac{1}{12}(K_{X_1} + 3r^*(K_X + L))(K_{X_1} + 2r^*(K_X + L))r^*(K_X + L)^2$$
$$\quad - \frac{1}{32}(6K_{X_1}L_1 + 9L_1^2)r^*(K_X + L)^2 + \frac{1}{24}(2K_{X_1} + 2r^*(K_X + L))r^*(K_X + L)^3$$
$$= -1 + h^1(\mathcal{O}_{X_1}) + \frac{7}{6}(K_X + L)^4 - \frac{41}{48}(K_X + L)^3 L - \frac{1}{96}(K_X + L)^2 L^2.$$

Hence by (23) we have

$$g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, L) - h^1(\mathcal{O}_X)$$
$$\geq \frac{7}{6}(K_X + L)^4 + \frac{31}{48}(K_X + L)^3 L - \frac{1}{96}(K_X + L)^2 L^2.$$

Here we note that by [19, Theorem 3.2 (i)] we have

(24) $$\qquad g_2(X, K_X + L, K_X + L) = h^0(3K_X + 2L) - 2h^0(2K_X + L)$$

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because X is rationally connected. If $g_2(X, K_X + L, K_X + L) \geq 0$, then $g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, L) - h^1(\mathcal{O}_X) \geq g_1(X, K_X + L, K_X + L, L) > 0$ because $h^1(\mathcal{O}_X) = 0$ and this contradicts (22). Therefore $g_2(X, K_X + L, K_X + L) < 0$ and we see from (24) that $h^0(2K_X + L) > 0$. Here we note that

$$(K_X + L)^3 L \;\; = \;\; (K_X + \frac{1}{2}L)(K_X + L)^2 L + \frac{1}{2}(K_X + L)^2 L^2.$$

Then

$$\frac{31}{48}(K_X + L)^3 L \;\; = \;\; \frac{31}{48}(K_X + \frac{1}{2}L)(K_X + L)^2 L + \frac{31}{96}(K_X + L)^2 L^2$$

$$\geq \;\; \frac{31}{96}(K_X + L)^2 L^2.$$

Therefore

$$g_2(X, K_X + L, K_X + L) + g_1(X, K_X + L, K_X + L, L) - h^1(\mathcal{O}_X)$$

$$\geq \frac{7}{6}(K_X + L)^4 + \frac{31}{48}(K_X + L)^3 L - \frac{1}{96}(K_X + L)^2 L^2$$

$$\geq \frac{7}{6}(K_X + L)^4 + \frac{5}{16}(K_X + L)^2 L^2$$

$$> 0.$$

This is also impossible.
Therefore we get $h^0(3(K_X + L)) > 0$. $\qquad\square$

By Theorems 3.3, 3.4 and Remark 2.3 we get the following corollary.

Corollary 3.1 $m_4^{\mathrm{NEF}}(\mathrm{SRE}) \leq m_4^{\mathrm{NEF}}(\mathrm{SRE})^+ \leq 3$.

Next we consider the case of $\kappa(X) \geq 0$.

Theorem 3.5 *Let* (X, L) *be a polarized variety of dimension 4. Assume that* (X, L) *satisfies the assumption* (SRE), $\kappa(X) \geq 0$ *and* $K_X + L$ *is nef. Then* $h^0(2(K_X + L)) \geq 1$.

Proof. We use Notation 3.1. Then $K_{X_1} + L_1 = r^*(K_X + L) + E_r$ holds, where E_r is an effective r-exceptional divisor. Hence for any positive integer m

$$h^0(m(K_{X_1} + L_1)) \;\; = \;\; h^0(mr^*(K_X + L) + mE_r)$$

$$= \;\; h^0(mr^*(K_X + L))$$

$$= \;\; h^0(m(K_X + L)).$$

Here we also note that

$$h^0(2K_{X_1} + 2L_1) \;\; = \;\; h^0(K_{X_1} + K_{X_1} + L_1 + L_1)$$

$$= \;\; h^0(K_{X_1} + r^*(K_X + L) + E_r + L_1)$$

$$\geq \;\; h^0(K_{X_1} + r^*(K_X + L) + L_1)$$

$$= \;\; h^0(r^*(K_X) + r^*(K_X + L) + r^*(L) + E_r)$$

$$\geq \;\; h^0(r^*(2K_X + 2L))$$

$$= \;\; h^0(2K_X + 2L).$$

Since $h^0(2K_{X_1} + 2L_1) = h^0(2K_X + 2L)$, we have $h^0(2K_{X_1} + 2L_1) = h^0(K_{X_1} + r^*(K_X + L) + L_1)$.

Then

(25)
$$h^0(2(K_X + L)) - h^0(K_X + L)$$
$$= h^0(2(K_{X_1} + L_1)) - h^0(K_{X_1} + L_1)$$
$$= h^0(K_{X_1} + r^*(K_X + L) + L_1) - h^0(K_{X_1} + L_1)$$
$$= g_3(X_1, r^*(K_X + L)) + g_2(X_1, r^*(K_X + L), L_1) - h^2(\mathcal{O}_{X_1})$$
$$= -\chi_3^H(X_1, r^*(K_X + L)) + \chi_2^H(X_1, r^*(K_X + L), L_1).$$

By using [14, (2.2.B)] and [16, Corollary 2.7] we have

$$-\chi_3^H(X_1, r^*(K_X + L))$$
$$= \frac{1}{24}(r^*(K_X + L))^4 + \frac{1}{12}K_{X_1}(r^*(K_X + L))^3$$
$$+ \frac{1}{24}(K_{X_1}^2 + c_2(X_1))(r^*(K_X + L))^2 + \frac{1}{24}K_{X_1}c_2(X_1)(r^*(K_X + L)),$$

and

$$\chi_2^H(X_1, r^*(K_X + L), L_1)$$
$$= \frac{1}{6}L_1(r^*(K_X + L))^3 + \frac{1}{4}(L_1)^2(r^*(K_X + L))^2 + \frac{1}{6}(L_1)^3(r^*(K_X + L))$$
$$+ \frac{1}{4}K_{X_1}(r^*(K_X + L) + L_1)(r^*(K_X + L))L_1 + \frac{1}{12}(K_{X_1}^2 + c_2(X_1))(r^*(K_X + L))L_1.$$

Hence

$$-\chi_3^H(X_1, r^*(K_X + L)) + \chi_2^H(X_1, r^*(K_X + L), L_1)$$
$$= \frac{1}{24}(r^*(K_X + L))^2 \left\{ (r^*(K_X + L))^2 + 2K_{X_1}r^*(K_X + L) + K_{X_1}^2 + 4L_1 r^*(K_X + L) \right.$$
$$+ 6L_1^2 + 6K_{X_1}L_1 \right\} + \frac{1}{24}c_2(X_1)(r^*(K_X + L)^2 + K_{X_1}r^*(K_X + L) + 2L_1 r^*(K_X + L))$$
$$+ \frac{1}{24}(4L_1^3 r^*(K_X + L) + 6K_{X_1}L_1^2 r^*(K_X + L) + 2K_{X_1}^2 L_1 r^*(K_X + L))$$
$$= \frac{1}{24}(r^*(K_X + L))^2(K_{X_1} + r^*(K_X + L))^2 + \frac{5}{12}(r^*(K_X + L))^3 L_1$$
$$+ \frac{1}{24}c_2(X_1)r^*(K_X + L)(r^*(K_X + L) + K_{X_1} + 2L_1) + \frac{1}{12}r^*(K_X + L)^2 r^*(K_X + 2L)L_1.$$

Here we note that by Lemma 2.4 we have $c_2(X_1)E_r r^*(K_X + L) = 0$. We also note that by Lemma 2.3 $(r^*(K_X + L))^2(K_{X_1} + r^*(K_X + L))^2 = (r^*(K_X + L))^2(r^*(2K_X + L))^2$ holds.
Hence

(26)
$$-\chi_3^H(X_1, r^*(K_X + L)) + \chi_2^H(X_1, r^*(K_X + L), L_1)$$
$$= \frac{1}{24}(r^*(K_X + L))^2(r^*(2K_X + L))^2 + \frac{5}{12}(r^*(K_X + L))^3 r^*(L)$$
$$+ \frac{1}{24}c_2(X_1)r^*(K_X + L)(r^*(2K_X + 3L)) + \frac{1}{12}r^*(K_X + L)^2 r^*(K_X + 2L)r^*(L).$$

By setting $H_1 := 2K_X + 3L$, $H_2 := K_X + L$, $H := L$ and $s = 1$, and by applying [19, Theorem 4.2], we have

(27)
$$c_2(X_1)r^*(2K_X + 3L)r^*(K_X + L)$$
$$\geq -\frac{3}{4}K_{X_1}r^*(L)r^*(K_X + L)r^*(2K_X + 3L) - \frac{3}{8}(r^*(L))^2 r^*(K_X + L)r^*(2K_X + 3L).$$
$$= -\frac{3}{4}r^*(K_X)r^*(L)r^*(K_X + L)r^*(2K_X + 3L) - \frac{3}{8}(r^*(L))^2 r^*(K_X + L)r^*(2K_X + 3L).$$

By (25), (26) and (27), we get

$$(28) \quad h^0(2(K_X + L)) - h^0(K_X + L)$$

$$= -\chi_3^H(X_1, r^*(K_X + L)) + \chi_2^H(X_1, r^*(K_X + L), L_1)$$

$$\geq \frac{1}{24}(K_X + L)^2(2K_X + L)^2 + \frac{5}{12}(K_X + L)^3 L$$

$$- \frac{1}{24}\left(\frac{3}{4}K_X L(K_X + L)(2K_X + 3L) + \frac{3}{8}L^2(K_X + L)(2K_X + 3L)\right)$$

$$+ \frac{1}{12}(K_X + L)^2(K_X + 2L)L$$

$$= \frac{1}{192}(K_X + L)\left\{8(K_X + L)(2K_X + L)^2 + 80(K_X + L)^2 L\right.$$

$$\left. - 6K_X L(2K_X + 3L) - 3L^2(2K_X + 3L) + 16(K_X + L)(K_X + 2L)L\right\}$$

$$= \frac{1}{192}(K_X + L)\left\{32K_X(K_X + 2L)^2 + 20K_X(K_X + 2L)L + 56(K_X + L)L^2 + 55L^3\right\}$$

$$\geq \frac{1}{192}(K_X + L)\left\{56(K_X + L)L^2 + 55L^3\right\}$$

$$\geq \frac{111}{192}.$$

Therefore we get the assertion. $\qquad\square$

Theorem 3.6 *Let (X, L) be a polarized variety of dimension 4. Assume that (X, L) satisfies the assumption* (SRE), $K_X + L$ *is nef and $\kappa(X) \geq 0$. Then for every integer m with $m \geq 2$*

$$h^0(m(K_X + L)) \geq \frac{(m-1)(m-2)(m^2 + 3m + 6)}{12} + 1.$$

Proof. We use Notation 3.1. As in the proof of Theorem 3.5, we have $h^0(m(K_{X_1} + L_1)) = h^0(m(K_X + L))$ for any positive inetger m. On the other hand,

$$\begin{aligned}
h^0(m(K_{X_1} + L_1)) &= h^0(K_{X_1} + (m-1)(K_{X_1} + L_1) + L_1) \\
&= h^0(K_{X_1} + (m-1)r^*(K_X + L) + r^*(L)) \\
&= h^0(K_{X_1} + (m-2)r^*(K_X) + (m-1)r^*(L) + r^*(K_X + L)), \\
h^0((m-1)(K_{X_1} + L_1)) &= h^0(K_{X_1} + (m-2)r^*(K_X) + (m-1)r^*(L)).
\end{aligned}$$

Let $F(t) := h^0(t(K_X + L)) - h^0((t-1)(K_X + L))$. Then by Theorem 2.1 we have

$$\begin{aligned}
F(t) &= h^0(K_{X_1} + (t-2)r^*(K_X) + (t-1)r^*(L) + r^*(K_X + L)) \\
&\quad - h^0(K_{X_1} + (t-2)r^*(K_X) + (t-1)r^*(L)) \\
&= g_3(X_1, r^*(K_X + L)) + g_2(X_1, (t-2)r^*(K_X + L) + r^*(L), r^*(K_X + L)) - h^2(\mathcal{O}_X).
\end{aligned}$$

Hence we have

$$\begin{aligned}
F(t) - F(t-1) &= g_2(X_1, (t-2)r^*(K_X + L) + r^*(L), r^*(K_X + L)) \\
&\quad - g_2(X_1, (t-3)r^*(K_X + L) + r^*(L), r^*(K_X + L)).
\end{aligned}$$

On the other hand by Proposition 2.1

$$\begin{aligned}
&g_2(X_1, (t-2)r^*(K_X + L) + r^*(L), r^*(K_X + L)) \\
&= g_2(X_1, (t-3)r^*(K_X + L) + r^*(L), r^*(K_X + L)) + g_2(X_1, r^*(K_X + L), r^*(K_X + L)) \\
&\quad + g_1(X_1, (t-3)r^*(K_X + L) + r^*(L), r^*(K_X + L), r^*(K_X + L)) - h^1(\mathcal{O}_{X_1}).
\end{aligned}$$

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Hence

$$F(t) - F(t-1) = g_1(X_1, (t-3)r^*(K_X + L) + r^*(L), r^*(K_X + L), r^*(K_X + L))$$
$$+ g_2(X_1, r^*(K_X + L), r^*(K_X + L)) - h^1(\mathcal{O}_{X_1}).$$

Here we note that $(K_X + L)L^3 \geq 2$ because $(K_X + L)L^3$ is positive and even. Moreover since

$$(K_X + L)^4 \geq (K_X + L)^3 L \geq (K_X + L)^2 L^2 \geq (K_X + L)L^3 \geq 2,$$

we have

$$g_1(X_1, (t-3)r^*(K_X + L) + r^*(L), r^*(K_X + L), r^*(K_X + L))$$
$$= 1 + \frac{t}{2}((t-3)K_X + (t-2)L)(K_X + L)^3$$
$$= 1 + \frac{t(t-3)}{2}(K_X + L)^4 + \frac{t}{2}L(K_X + L)^3$$
$$\geq 1 + t(t-3) + t$$
$$= (t-1)^2.$$

By [19, Corollary 4.1] we have $g_2(X_1, r^*(K_X + L), r^*(K_X + L)) \geq h^1(\mathcal{O}_{X_1})$. Hence

$$F(t) - F(t-1) \geq (t-1)^2$$

and

$$h^0(k(K_X + L)) - h^0((k-1)(K_X + L))$$
$$= F(k)$$
$$\geq (k-1)^2 + \cdots + 2^2 + F(2)$$
$$= \frac{k(k-1)(2k-1)}{6} - 1 + F(2).$$

Claim 3.6 $h^0(2(K_X + L)) - h^0(K_X + L) \geq 0.$

Proof. If $h^0(K_X + L) \geq 1$, then by Lemma 2.1 we get $h^0(2(K_X + L)) - h^0(K_X + L) \geq h^0(K_X + L) - 1 \geq 0$. Hence $h^0(2(K_X + L)) - h^0(K_X + L) \geq 0$. If $h^0(K_X + L) = 0$, then by Theorem 3.5 we get $h^0(2(K_X + L)) - h^0(K_X + L) \geq 1$. So we get the assertion. $\square$

By Claim 3.6, $F(2) \geq 0$. Hence

$$h^0(k(K_X + L)) - h^0((k-1)(K_X + L)) \geq \frac{k(k-1)(2k-1)}{6} - 1.$$

Therefore

$$h^0(m(K_X + L)) \geq h^0(2(K_X + L)) + \sum_{k=3}^{m}\left\{\frac{k(k-1)(2k-1)}{6} - 1\right\}$$
$$= h^0(2(K_X + L)) + \frac{1}{12}m^2(m^2 - 1) - (m-1)$$
$$\geq 1 + \frac{1}{12}m^2(m^2 - 1) - (m-1)$$
$$= \frac{(m-1)(m-2)(m^2 + 3m + 6)}{12} + 1.$$

We get the assertion of Theorem 3.6. $\square$

4   The case of $\kappa(K_X + L) \geq 0$

In this section, we consider the case of $\kappa(K_X + L) \geq 0$ in general. First we review the adjunction theory of Beltrametti-Sommese and Fujita, which will be used later.

Theorem 4.1 *Let $(X, \mathcal{L})$ be a polarized manifold with* $\dim X = n \geq 3$. *Then $(X, \mathcal{L})$ is one of the following types.*

(1) $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.

(2) $(\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.

(3) *A scroll over a smooth projective curve.*

(4) *A Del Pezzo manifold.*

(5) *A quadric fibration over a smooth curve.*

(6) *A scroll over a smooth projective surface.*

(7) *Let $(M, \mathcal{A})$ be a reduction of $(X, \mathcal{L})$.*

    (7.1) $n = 4$, $(M, \mathcal{A}) = (\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2))$.

    (7.2) $n = 3$, $(M, \mathcal{A}) = (\mathbb{Q}^3, \mathcal{O}_{\mathbb{Q}^3}(2))$.

    (7.3) $n = 3$, $(M, \mathcal{A}) = (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3))$.

    (7.4) $n = 3$, M is a $\mathbb{P}^2$-bundle over a smooth curve C and for any fiber F' of it, $(F', \mathcal{A}|_{F'}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

    (7.5) $K_M \sim -(n-2)\mathcal{A}$, that is, $(M, \mathcal{A})$ is a Mukai manifold.

    (7.6) $(M, \mathcal{A})$ is a Del Pezzo fibration over a smooth curve.

    (7.7) $(M, \mathcal{A})$ is a quadric fibration over a normal surface.

    (7.8) $n \geq 4$ and $(M, \mathcal{A})$ is a scroll over a normal projective variety of dimension 3.

    (7.9) $K_M + (n-2)\mathcal{A}$ is nef and big.

Proof.   See [4, Proposition 7.2.2, Theorem 7.2.4, Theorem 7.3.2, Theorem 7.3.4, and Theorem 7.5.3]. See also [10, Chapter II, (11.2), (11.7), and (11.8)]. $\qquad\square$

Remark 4.1 Let $(X, \mathcal{L})$ be a polarized manifold with $\dim X = n \geq 3$.

(1) $\kappa(K_X + (n-2)\mathcal{L}) = -\infty$ if and only if $(X, \mathcal{L})$ is one of the types from (1) to (7.4) in Theorem 4.1.

(2) $\kappa(K_X + (n-2)\mathcal{L}) = 0$ if and only if $(X, \mathcal{L})$ is (7.5) in Theorem 4.1.

(3) $\kappa(K_X + (n-2)\mathcal{L}) \geq 1$ if and only if $(X, \mathcal{L})$ is one of the types from (7.6) to (7.9) in Theorem 4.1.

Definition 4.1 Let $(X, \mathcal{L})$ be a polarized manifold of dimension $n \geq 3$, and let $(M, \mathcal{A})$ be a reduction of $(X, \mathcal{L})$. Assume that $K_M + (n-2)\mathcal{A}$ is nef and big. Then for large $m \gg 0$ the morphism $\varphi : M \to W$ associated to $|m(K_M + (n-2)\mathcal{A})|$ has connected fibers and normal image W. Then we note that there exists an ample line bundle $\mathcal{K}$ on W such that $K_M + (n-2)\mathcal{A} = \varphi^*(\mathcal{K})$. Let $\mathcal{D} := (\varphi_* \mathcal{A})^{\vee\vee}$, where $^{\vee\vee}$ denotes the double dual. Then the pair $(W, \mathcal{D})$ together with φ is called the *second reduction of $(X, \mathcal{L})$*.

Remark 4.2 (1) If $K_M + (n-2)\mathcal{A}$ is nef and big but not ample, then φ is equal to the nef value morphism of $\mathcal{A}$.

(2) If $K_M + (n-2)\mathcal{A}$ is ample, then φ is an isomorphism.

(3) If $n \geq 4$, then W has isolated terminal singularities and is 2-factorial. Moreover if n is even, then X is Gorenstein (see [4, Proposition 7.5.6]).

Here we consider a characterization of $(X, \mathcal{L})$ with $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$. We note that $\kappa(K_X + (n-1)\mathcal{L}) = -\infty$ (resp. $\kappa(K_X + (n-2)\mathcal{L}) = -\infty$) if and only if $(X, \mathcal{L})$ is one of the types from (1) to (3) (resp. from (1) to (7.4)). Here we consider the case where $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$. If $(X, \mathcal{L})$ is one of the types from (1) to (7.8), then $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$ holds. So we assume that $K_M + (n-2)\mathcal{A}$ is nef and big. Then there exist a normal projective variety W with only 2-factorial isolated terminal singularities, a birational morphism $\phi_2 : M \to W$ and an ample line bundle $\mathcal{K}$ on W such that $K_M + (n-2)\mathcal{A} = (\phi_2)^*(\mathcal{K})$. Let $\mathcal{D} := (\phi_2)_*(\mathcal{A})^{\vee\vee}$. Then $\mathcal{D}$ is a 2-Cartier divisor on W and $\mathcal{K} = K_W + (n-2)\mathcal{D}$ (see [4, Lemma 7.5.8]). Then the pair $(W, \mathcal{D})$ is the second reduction of $(X, \mathcal{L})$ (see Definition 4.1). Here we remark that if $K_M + (n-2)\mathcal{A}$ is ample, then $(W, \mathcal{K}) \cong (M, K_M + (n-2)\mathcal{A})$.

Then the following properties hold:

(1) $\kappa(K_X + (n-3)\mathcal{L}) = \kappa(K_W + (n-3)\mathcal{K})$ holds [4, Corollary 7.6.2].

(2) $(n-2)(K_W + (n-3)\mathcal{D}) = K_W + (n-3)\mathcal{K}$ and $K_M + (n-3)\mathcal{A} = \phi_2^*(K_W + (n-3)\mathcal{D}) + \Delta$ for an exceptional effective $\mathbb{Q}$-Cartier divisor Δ of ϕ_2. Therefore

$$
\begin{aligned}
m(n-2)(K_X + (n-3)\mathcal{L}) &= m(n-2)\phi_1^*(K_M + (n-3)\mathcal{A}) + E_1 \\
&= m(n-2)\phi_1^*(\phi_2^*(K_W + (n-3)\mathcal{D})) + E_1 + m(n-2)\phi_1^*\Delta \\
&= m\phi_1^* \circ \phi_2^*(K_W + (n-3)\mathcal{K}) + E_1 + m(n-2)\phi_1^*\Delta.
\end{aligned}
$$

(Here $\phi_1 : X \to M$ is a reduction of $(X, \mathcal{L})$ and E_1 is a ϕ_1-exceptional effective divisor.)

(3) $h^0((n-2)m(K_X + (n-3)\mathcal{L})) = h^0(m(K_W + (n-3)\mathcal{K}))$ for every integer m with $m \geq 1$.

Moreover if $n \geq 4$, then there exists a normal factorial projective variety $M^\sharp$ with only isolated terminal singularities and birational morphisms $\phi_2^\sharp : M \to M^\sharp$ and $\psi : M^\sharp \to W$ such that $\phi_2 = \psi \circ \phi_2^\sharp$. Then $M^\sharp$ is called the *factorial stage* (see [4, 7.5.7 Definition-Notation] or [11, (2.4) Theorem]).

Here we consider a classification of $(X, \mathcal{L})$ with $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$ and $n \geq 4$. First we note that if $(X, \mathcal{L})$ is one of the types from (1) to (7.8), then we see that $\kappa(K_X + (n-3)\mathcal{L}) = -\infty$. So we may assume that $K_M + (n-2)\mathcal{A}$ is nef and big. Then there exists the second reduction $(W, \mathcal{D})$ of X. Here we use notation in Definition 4.1. If $\tau(\mathcal{K}) \leq n-3$, then by above we see that $\kappa(K_X + (n-3)\mathcal{L}) \geq 0$. (Here $\tau(\mathcal{K})$ denotes the nef value of $\mathcal{K}$.) So we may assume that $\tau(\mathcal{K}) > n-3$.

Here we consider the case of $n = 4$. In this case $M^\sharp$ and W are Gorenstein (see [4, Proposition 7.5.6 and 7.5.7 Definition-Notation]). Then by the proof of [11, Section 4] we see that $(W, \mathcal{K})$ or $M^\sharp$ is one of the types in [11, (4.∞)]. If $(W, \mathcal{K})$ or $M^\sharp$ is either (4.2), (4.4.0), (4.4.1), (4.4.2), (4.6.0.0), (4.6.0.1.0), (4.6.0.2.1), (4.6.1), (4.7) or (4.8.0) in [11, (4.∞)], then we see that $\kappa(K_X + \mathcal{L}) = -\infty$.

Assume that $(W, \mathcal{K})$ is the type (4.4.4) in [11, (4.∞)]. Then we note that $\tau(\mathcal{K}) = 3$ and there exist a normal Gorenstein projective variety W_2, an ample line bundle $\mathcal{K}_2$ on W_2 and a birational morphism $\mu : W \to W_2$ such that μ is the simultaneous contraction to distinct smooth points of divisors $E_i \cong \mathbb{P}^3$ such that $E_i \subset \mathrm{reg}(W)$, $E_i|_{E_i} \cong \mathcal{O}_{\mathbb{P}^3}(-1)$, $K_W + 3\mathcal{K} = \mu^*(K_{W_2} + 3\mathcal{K}_2)$ and $K_{W_2} + 3\mathcal{K}_2$ is ample, that is, $\tau(\mathcal{K}_2) < 3$. Moreover we infer that W_2 has the same singularities as W by above. Since $E_i \subset \mathrm{reg}(W)$, we have $\psi^{-1}(E_i) \cong E_i$ by the definition of ψ. Hence there exist a normal Gorenstein projective variety $W_2^\sharp$ and birational morphisms $\mu^\sharp : M^\sharp \to W_2^\sharp$ and $\psi^\sharp : W_2^\sharp \to W_2$ such that $\mu \circ \psi = \psi^\sharp \circ \mu^\sharp$. We note that $\mu^\sharp : M^\sharp \to W_2^\sharp$ is the contraction of

$\psi^{-1}(E_i)$ and $W_2^\sharp$ has the same singularities as $M^\sharp$. $(W_2, \mathcal{K}_2)$ is a reduction of $(W, \mathcal{K})$ and is called the $2\frac{1}{2}$ *reduction of* $(W, \mathcal{K})$ in [3, (2.2) Theorem-Definition]. We also note that $h^j(\mathcal{O}_X) = h^j(\mathcal{O}_M) = h^j(\mathcal{O}_W) = h^j(\mathcal{O}_{W_2}) = h^j(\mathcal{O}_{M^\sharp}) = h^j(\mathcal{O}_{W^\sharp})$. For this $\psi^\sharp : W_2^\sharp \to W_2$ and $(W_2, \mathcal{K}_2)$, we can apply the same argument as in [11, Section 4]. If $\tau(\mathcal{K}_2) \leq 1$, then we can prove that $\kappa(K_X + \mathcal{L}) \geq 0$. So we assume that $\tau(\mathcal{K}_2) > 1$. Then $(W_2, \mathcal{K}_2)$ is either (4.6.0.0), (4.6.0.1.0), (4.6.0.2.1), (4.6.1), (4.6.4), (4.7) or (4.8.0) in [11, $(4.\infty)$].

If $(W_2, \mathcal{K}_2)$ is either (4.6.0.0), (4.6.0.1.0), (4.6.0.2.1), (4.6.1), (4.7) or (4.8.0) in [11, $(4.\infty)$], then we see that $\kappa(K_X + \mathcal{L}) = -\infty$.

If $(W_2, \mathcal{K}_2)$ is the type (4.6.4) in [11, $(4.\infty)$], then by the same argument as in [11, Section 4] we see that there exist a normal Gorenstein projective variety W_3, an ample line bundle $\mathcal{K}_3$ on W_3 and a birational morphism $\mu_2 : W_2 \to W_3$ such that W_3 has the same singularities as W_2, $K_{W_2} + 2\mathcal{K}_2 = \mu_2^*(K_{W_3} + 2\mathcal{K}_3)$ and $K_{W_3} + 2\mathcal{K}_3$ is ample, that is, $\tau(\mathcal{K}_3) < 2$. Here we note that $\kappa(K_X + \mathcal{L}) = \kappa(K_{W_2} + \mathcal{K}_2) = \kappa(K_{W_3} + \mathcal{K}_3)$.

If $\tau(\mathcal{K}_3) \leq 1$, then $\kappa(K_X + \mathcal{L}) = \kappa(K_{W_3} + \mathcal{K}_3) \geq 0$.

If $\tau(\mathcal{K}_3) > 1$, then $(W_3, \mathcal{K}_3)$ is either (4.7) or (4.8.0) in [11, $(4.\infty)$] by the same argument as in [11, Section 4] and we have $\kappa(K_X + \mathcal{L}) = \kappa(K_{W_3} + \mathcal{K}_3) = -\infty$.

By the above argument, we get the following:

Theorem 4.2 *Let* $(X, \mathcal{L})$ *be a polarized manifold of dimension* $n = 4$.

(1) $\kappa(K_X + \mathcal{L}) \geq 0$ *if and only if there exist a normal Gorenstein projective variety* W_3 *with only isolated terminal singularities, an ample line bundle* $\mathcal{H}_3$ *on* W_3, *and a birational morphism* $\Phi : X \to W_3$ *such that* $\tau(\mathcal{H}_3) \leq 1$ *and* $h^0(2m(K_X + \mathcal{L})) = h^0(m(K_{W_3} + \mathcal{H}_3))$ *for every positive integer* m.

(2) $\kappa(K_X + \mathcal{L}) = -\infty$ *if and only if* $(X, \mathcal{L})$ *satisfies one of the following:*

(2.1) $(X, \mathcal{L})$ *is either* (1), (2), (3), (4), (5), (6), (7.1), (7.5), (7.6), (7.7) *or* (7.8) *in Theorem* 4.1.

(2.2) *There exist a normal projective variety* W_3 *with only isolated terminal singularities, an ample line bundle* $\mathcal{H}_3$ *on* W_3, *and a birational morphism* $\Phi : X \to W_3$ *such that* $(W_3, \mathcal{H}_3)$ *is either* (4.2), (4.4.0), (4.4.1), (4.4.2), (4.6.0.0), (4.6.0.1.0), (4.6.0.2.1), (4.6.1), (4.7) *or* (4.8.0) *in* [11, $(4.\infty)$].

By Theorems 3.2, 3.3, 3.4, 3.6 and 4.2, we get Theorem 1 in the Introduction. As a corollary, we have the following.

Corollary 4.1 $m_4(\text{SM}) \leq 6$. *Here the assumption* (SM) *is the following.*
(SM)*:* X *is smooth.*

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